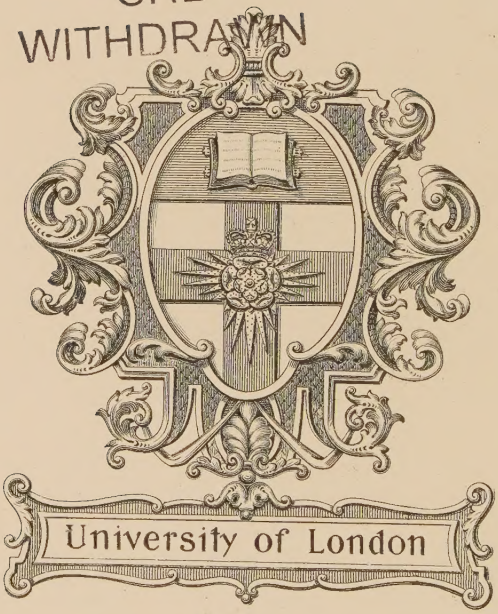
PRIVATE
PB [Depository Mathematical Society.]

SHL
WITHDRAWN



Digitized by the Internet Archive
in 2026

https://archive.org/details/london-mathematical-society-proceedings_1882-1883_14

PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XIV.

FROM NOVEMBER, 1882, TO NOVEMBER, 1883.

LONDON:
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

LONDON:

PRINTED BY C. F. HODGSON AND SON,

GOUGH SQUARE, FLEET STREET.

BIBL.
LONDIN.
UNIV.

PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XIV.

NINETEENTH SESSION, 1882-83.

November 9th, 1882.

ANNUAL GENERAL MEETING, held at 22 Albemarle Street.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

The Treasurer (C. W. Merrifield, F.R.S.) read his Report, which, on the motion of Mr. W. J. Curran Sharp, seconded by Mr. C. Leudesdorf, was accepted.

At the request of the Chairman, Mr. J. Hammond consented to act as Auditor.

The Secretaries' Report was read and adopted.

From the Report of the Secretaries, it appeared that the number of members during the past Session had increased from 159 to 165, of whom 54 are Compounders; and that the Society had lost by death Prof. W. Stanley Jevons, F.R.S., and Dr. William Robertson, F.R.S.E.

The communications made to the Society during the past Session had been as follows:—

"Note on the Limit to the Number of different Proper Fractions whose Denominators are less than x, where x is large:" Messrs. Jenkins & Merrifield.

"Note on Landen's Theorem:" Prof. Cayley.

"On the Oscillations of a Viscous Spheroid:" Prof. H. Lamb.

"A Geometrical Representation of a System of two Binary Cubics and their Associated Forms:" Mr. W. R. W. Roberts.

"On the Infinitesimal Bending of Surfaces of Revolution:" Lord Rayleigh.

"On Tangents to a Cubic forming a Pencil in Involution:" Mr. R. A. Roberts.

"On the Polar Planes of Four Quadrics:" Dr. W. Spottiswoode.

"On some Forms of Cubic Determinants:" Mr. R. F. Scott.

"On the Flow of a Viscous Fluid through a Pipe or Channel:" Mr. A. G. Greenhill.

"On the Covariant Locus of the Vertex of a Pencil of Tangents to a Cubic in Involution:" Mr. J. J. Walker.

"The Invariants of a certain Orthogonal Transformation, with special reference to their use in the Theory of the Strains and Stresses of an Elastic Solid:" Mr. W. J. C. Sharp.

"Some Elliptic Function Formulæ:" Rev. M. M. U. Wilkinson.

"On the Calculation of Symmetric Functions:" Mr. J. Hammond.

"Complete Determination of the Real Foci and of the Vector Equation of the Pedal of a given Ellipse with respect to any proposed Point:" Prof. Wolstenholme.

"On certain Quartic Curves having a Tacnode at Infinity at which the Line at Infinity is the Multiple Tangent:" Mr. H. M. Jeffery.

"Note on a Geometrical Theorem:" Messrs. H. M. Taylor and R. C. Rowe.

"Systems of Formulæ for the sn, cn, and dn of $u+v+w$:" Prof. W. Woolsey Johnson.

"On the Addition Equations for the Elliptic and Theta Functions of the Sum of n Arguments:" Rev. M. M. U. Wilkinson.

"Proofs of the Addition Theorems for Elliptic Integrals of the second kind, and of Fagnani's Theorem:" Mr. J. J. Walker.

"The Algebraic Solution of the Modular Equation for the Septic Transformation:" Mr. G. S. Ely.

"Note on the Condensation of Skew Determinants, which are partially Zero-Axial," and "On a Symmetric Determinant, connected with Lagrange's Interpolation Problem:" Mr. T. Muir.

"On the General Equation of the Second Degree referred to Tetrahedral Coordinates:" Rev. A. J. C. Allen.

"On certain Loci and Envelopes belonging to Triangles of given Form, inscribed in and circumscribed to a given Triangle:" Prof. Wolstenholme.

"On Binomial Biordinals:" Sir J. Cockle.

"On the Coordinates of a Plane Curve in Space:" Mr. H. W. Lloyd Tanner.

"On Polygons circumscribed about a Cuspidal Cubic:" Mr. R. A. Roberts.

"A Method of General Inversion due to M. Vanecek (of Prague):" Dr. Hirst.

"On a Formula relating to Elliptic Integrals of the Third Kind:" Prof. Cayley.

"Elementary Analytical Proof of Graves's and MacCullagh's Theorems, with an Extension of the Former:" Mr. J. Griffiths.

"Note on a System of Confocal Bicircular Quartics:" Mr. R. A. Roberts.

"On the Vibrations of an Elastic Sphere:" Prof. H. Lamb.

"On the Extension of certain Theories relating to Plane Cubics to Curves of any Deficiency:" Mr. A. Buchheim.

"On some Formulæ arising from the Differentiation of Elliptic Functions with regard to the Modulus:" Rev. M. M. U. Wilkinson.

Two Notes: (1) "A Definite Integral," (2) "Equation to a Director-circle of a Conic:" Prof. Wolstenholme.

"Theory of Orthoptic Loci:" Rev. Dr. Taylor.

Besides the above, a few minor communications were made.

The same Journals had been subscribed for as in the preceding Session, and the same exchanges made, with the addition of the "Reale Istituto Veneto di Scienze, Lettere ed Arti."

After the reading of the Report, the Chairman spoke upon the loss the Society had sustained by the untimely death of Prof. Jevons.

The Meeting then proceeded to the election of the new Council.

The Scrutators (Messrs. Leudesdorf and Terry) having examined the Balloting Lists, declared the following gentleman duly elected:—

President, Prof. O. Henrici, F.R.S.; Vice-Presidents, Sir J. Cockle, F.R.S., S. Roberts, F.R.S.; Treasurer, C. W. Merrifield, F.R.S.; Hon. Secs., M. Jenkins, M.A., and R. Tucker, M.A.; other Members, Prof. Cayley, F.R.S., E. B. Elliott, M.A., J. W. L. Glaisher, F.R.S., H. Hart, M.A., Dr. Hirst, F.R.S., Dr. Hopkinson, F.R.S., A. B. Kempe, F.R.S., R. F. Scott, M.A., Prof. H. J. S. Smith, F.R.S., and J. J. Walker, M.A.

Prof. Henrici having thanked the Members for his election, took the Chair, and called upon Mr. Roberts to read his Address, entitled "Remarks on Mathematical Terminology and the Philosophical Bearing of recent Mathematical Speculations concerning the Realities of Space." On the motion of Prof. Cayley, seconded by Mr. Merrifield, it was unanimously resolved that the Address be printed in the Proceedings of the Society.

Mr. W. M. Hicks, M.A, was admitted into the Society.

The following communications were made:—

"On In- and Circumscribed Polyhedra," by Prof. A. R. Forsyth.

"Note on Quartic Curves in Space," Dr. Spottiswoode, P.R.S.

"On Unicursal Twisted Quartics," Mr. R. A. Roberts.

"Note on the Derivation of Elliptic Function Formulæ from Confocal Conics," Mr. J. Griffiths.

"On the Explicit Integration of certain Differential Resolvents," Sir J. Cockle.

"On Compound Determinants," Mr. R. F. Scott.

The following presents were received :—

"Reale Istituto Lombardo di Scienze e Lettere," Rendiconti, Serie ii., Vol. xiii., 1880.

"Beiblätter zu den Annalen der Physik und Chemie ," Band 6, Stücke 7, 8, 9.

"Bulletin de la Société Mathématique de France," Tome x., No. 5.

"Crelle," Bd. xciii., H. 1 and 2.

"Atti della R. Accademia dei Lincei—Transunti," Vol. vi., Pts. 13 and 14 ed ultimo.

"The Wave-Lengths of some of the principal Fraunhofer Lines of the Solar Spectrum," by Dr. T. C. Mendenhall, 8vo ; Tokio, 1881.

"Instructions for Observing the Transit of Venus, Dec. 6th, 1882," 4to ; Washington, 1882.

"A Value of the Solar Parallax from Meridian Observations of Mars at the Opposition in 1877," by J. R. Eastman, 4to ; Washington, 1881.

"Number and Matter," by Wm. Sully Beebe, 4to ; New York.

"List of Foreign Correspondents of the Smithsonian Institution, corrected to Jan. 1882," 8vo ; Washington, 1882.

"Société de Géographie Commerciale de Bordeaux, Catalogue Spécial de l'Exposition," 8vo ; Bordeaux, 1882.

"Société de Géographie Commerciale de Bordeaux, Bulletin, No. 16," 24 Août, 1882.

"Proceedings of the Royal Society," No. 221, Vol. xxxiv.

"Mathematical Questions from Educational Times," Vol. xxxvii., 8vo ; London, 1882.

"Sur le Problème de Pfaff," par M. G. Darboux, 8vo ; Paris, 1882.

"Educational Times," August to November, 1882.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," xviii. to xxxviii., 8vo ; Berlin, 1882.

"Bulletin Astronomique et Météorologique de l'Observatoire Impérial de Rio de Janeiro," Nos. 3, 5, 6, 8.

"Bulletin des Sciences Mathématiques et Astronomiques—Table des Matières et Noms d'Auteurs," last part of Tome 5, four parts (March, April, May, and June) of Tome 6.

"The Physical Society of London—Proceedings," Vol. v., Pt. ii., July to September, 1882.

"Report on Recent Progress in Hydrodynamics," by W. M. Hicks ; Part ii., Special Problems.

"Jahrbuch über die Fortschritte der Mathematik," &c., Band. xii., Heft 2, Jahrgang 1880.

"Introduzione Storica alla Teoria delle Funzioni Ellittiche," per il Prof. G. Bellacchi," Fasc. i. and ii., 8vo ; Firenze, 1881 and 1882.

"Alhazen's Problem," by Marcus Baker, 4to (from "American Journal of Mathematics, Vol. iv., No. 4).

Remarks on Mathematical Terminology, and the Philosophic Bearing of recent Mathematical Speculations concerning the Realities of Space. By Samuel Roberts, F.R.S.

[*Read Nov. 9th*, 1882.]

On ceasing to be your President, I am privileged to address you in a less technical manner than usual, and I take the opportunity, in the first place, to congratulate you on our satisfactory progress. We now commence our nineteenth Session, and are well able to judge whether our Society has filled a place which needed to be filled, and is likely to be useful. There is, I think, no reason for apprehension on this head. It may be affirmed, without undue presumption, that many truths, first published to the world in our Proceedings, will long remain a valued part of the working capital of mathematicians. But absolute utility is by no means to be measured solely by special results entitled to rank among the heirlooms of science. If it were so, the mass of mathematical literature formed in a single century might well daunt those who know how much laborious thought is implied by communications of even small extent and unambitious character. Comparatively little of that formidable accumulation can, at the present day, be studied in detail. Many mathematicians of unquestionable force are almost forgotten in the hurry of modern thought, and innumerable results of their labours have, as it were, slipped through the sieve of history. The residue of permanent doctrine, of theorems and formulæ, which confer an immortality, very imperfectly represents what we owe to our predecessors. Their unremembered work was justified by suggestiveness and originality, which stimulated the minds of contemporaries, by aid rendered to new comers, destined perhaps to eclipse their masters, and by the ever-growing generalisation of ideas leading up to more comprehensive theories. In estimating our usefulness, gentlemen, we are entitled to take account of considerations of this kind. Those of our members who are near the outset of their career are especially to be congratulated on the existence of this Society. We offer to every mathematical *débutant*, academic or non-academic, an opportunity of securing genial appreciation of his work, of coming into contact with kindred minds, and benefiting by the suggestions of those who have been longer in the field. I venture, then, to congratulate you on our firm establishment.

We have, indeed, been recently enabled to take a step which, to some extent, marks maturity. I have the pleasure to state that your Council has found itself in a position to arrange for the recognition, in a tangible form, of services to mathematical science. You are aware that, by the courtesy and goodwill of the De Morgan Memorial

Committee, we have been put in possession, by the medium of trustees, of a sum of money, and an engraved die of suitable design. With the aid of some additional contributions by our members, a sufficient fund has been secured to enable us to award a gold medal, of the value of ten pounds, at intervals of three years. It is singularly appropriate that this mark of distinction should be associated with the name of our first President.

Your executive were anxious to determine the most advantageous conditions of such an award. It was generally felt that those conditions should be as little restrictive as possible; and that, having regard to the cosmopolitan character of our science, and to the great value which must attach to our cordial co-operation with foreign men of science, the award should be made without regard to nationality. The first selection will be made in time for awarding the medal at the annual general meeting in November, 1884. I confidently believe that this distinction, modest though it be in material value, will be regarded by its recipients as a great encouragement.

And now, before I have quite exhausted your patience, I ought to pass to subjects more closely allied to our special pursuits. Yet, as to this, I feel no small difficulty. I have the honour to address gentlemen who, for the most part, have benefited by the congenial associations and scientific air of the ancient universities,—advantages which I appreciate all the more because, by a policy of the past, they were denied to myself.

I am aware that the best thoughts of an isolated thinker may be the platitudes of an academy, and it becomes me to avoid a didactic tone in speaking to those whose experience has been so much richer than my own. I shall merely submit to your judgment some suggestions relating to the form rather than the substance of mathematical literature.

I think it quite possible for this Society to exercise a direct beneficial influence on mathematical style. Terminology and nomenclature must probably always be subject to natural growth; but it does not seem beyond the scope of our associated efforts to stimulate, and at the same time control, that growth in a sensible degree. We are not, I am afraid, remarkable among nations for excellencies of style and exposition. There are some predisposing causes amongst us which tend to impair mathematical composition on its literary side. We are a practical people. If we reach a valuable result, the chances are that we shall not care too much about the setting. Our young mathematicians find their path beset with competitive examinations, little calculated to advance the claims of such unremunerative things as form and style. High authorities testify to the correctness of both form and style exhibited by exceptional men in those ordeals. This

encouraging circumstance, however, in no way disproves a generally injurious tendency. In the next place, I may mention, as a disturbing influence, the high speed at which scientific work is going on. It is very well to say—

> "Da spatium tenuemque moram, male cuncta ministrat
> Impetus;"

if results are to be useful, they must be promptly promulgated. Lastly, I am convinced that many papers of great merit are seriously compromised by the sheer modesty of authors, who economise space, and spare the printer at the cost of obscurity.

> "For brevity is very good,
> When w'are or are not understood,"

is not a politic maxim.

Cannot we do something to counteract these influences? Mathematical style has more than æsthetic importance. The frippery of rhetoric we can dispense with, but there are three elements of primary moment—arrangement, terminology, and notation.

The first of these must be mainly left to the discretion of a trained scientific mind. The case of terminology is somewhat different. Your memories will readily supply historical facts which show that nomenclature and terminology in Mathematics have always been very important, and, at the same time, difficult matters. A slight survey of mathematical history makes us aware that there has been an enormous waste of word-making power. To some extent, this could hardly be avoided. Terminology is often a part of the scaffolding, removable when its purpose is fulfilled.

Looking back less than two centuries, we find more cause for wonder that many technical terms now obsolete should ever have been in vogue, than that they have ceased to be so. Take, for example, the doctrine of ratios. The remark of Dr. Wallis is naïve. "I should choose to say," he writes, "*As* 30 *to* 7, *or as* 7 *to* 31, rather than *Quadrupla super bi-partiens septimas*, or *Subquadrupla super tri-partiens septimas*." An impracticable technicality was a last tribute of our science to the scholastic spirit. Nevertheless, special terminology cannot be dispensed with in a progressive science. We have to reduce a necessary evil to its smallest proportions.

To do this, we should stringently apply the law of parsimony. When I think of those classical phrases—"the ϑ-functions," the "q-products"—I am thankful for their simplicity. If we are compelled by "*force majeure*" to innovate, our new terms should be formed in accordance with the genius of the language, with recognised rules of derivation, and, especially, with euphony. This is a more difficult business than

might be supposed. I am inclined to think that a chapter on Terminology and Nomenclature in relation to Mathematics, would be a useful and interesting feature of an advanced text-book. For, although it is often possible, and in a high degree advantageous, to appropriate idiomatic words, having some fit association or analogy, scientific terms are, with us, chiefly of Latin or Greek origin. On this account, if on no other, a certain amount of classical attainment is of undeniable utility to a scientific man. Modern tendencies, however, lead to the conclusion that the classical scholar and the mathematician will, more seldom than heretofore, be found united in the same person. Nor can it be expected that specialists will disentangle for themselves the rules of word-formation with entire success. I believe grammarians complain that science has introduced into the language many words of eccentric derivation. So much depends on the wise use of our inventive faculty in relation to new language, that the subject seems worthy of the particular attention of our members.

I cannot well omit some few words concerning Notation. Of course, notation, or algorithm, is of more intrinsic importance than nomenclature and terminology. Its history has been, as you know, very remarkable, and the actual course of events is difficult to reconcile with *à priori* principles. "It is very singular," says Hallam, "that discoveries of the greatest convenience, and, apparently, not above the ingenuity of a village schoolmaster, should have been overlooked by men of extraordinary acuteness, like Tartaglia, Cardan, and Ferrari; and hardly less so that, by dint of that acuteness, they dispensed with the aid of these contrivances, in which we suppose that so much of the utility of algebraical expression consists." In times much nearer our own, you know how injuriously the status of English mathematicians was affected by a too faithful adherence to the fluxional calculus and synthetical form.

Those, no doubt, were palmy days, when a reputation could be made by the invention of a new symbol. On the other hand, we have the advantage of being fully alive to the importance and difficulties of algorithm. Its principles have been studied in connection with the Calculus of operations, and some of the most remarkable developments of our century—as Quaternions and Multiple Algebra—are due to the study of symbols in themselves.

There is something, too, to be said on the artistic side. It has been remarked that a printed page is, or ought to be, a work of art. This is peculiarly true of mathematical printing. The choice of symbols is often an occasion for the exercise of taste and discretion. Much has been done by greater attention to symmetry, and the use of the determinant notation, and still the limits of perfection are far off. My excuse for troubling you with these obvious reflections is, that

minutiæ of this kind are apt to be overlooked amid larger aims, and it has happened, not unfrequently, that the attractiveness of mathematical inquiries has been marred, and the difficulty in no small degree enhanced, by defects of exposition, which a little extra "limæ labor" would remove.

The opportunities for interchange of thought which our meetings afford might, I think, be utilised to establish new terms and symbols, on a better basis, and with a better chance of permanency.*

I shall venture further to touch on a subject which is also mainly a question of language—at least, in the point of view from which I wish now to regard it.

We have hitherto, in our Proceedings, laudably held aloof from metaphysical discussions. Within the borders of our science, a neutral plateau is found, where minds of different philosophical temperament can harmoniously cooperate. This neutrality is precious, and should not be compromised by the intrusion of controversies, which have ever been active at the outskirts, into interior regions hitherto uninvaded.

On rare occasions, dissensions of a metaphysical order are historically prominent within the technical domain, but the citadels of the pure science have usually been respected. With an instinctive prudence, mathematicians have accepted in their technical works current phraseology, whenever this could be done consistently with mathematical cogency. Thus, they have spoken of time, space, force, and motion, while putting aside questions relating to the origin of these notions. This has not been from deficiency of philosophical ability, for it is difficult to name mathematicians of leading rank who have not been metaphysicians too, from natural bent or force of circumstances.

Misgivings as to the permanence of our repose have been occasioned by modern speculations, which have culminated in the doctrines of hyperbolic, elliptic, and n-dimensioned spaces.

No person, who has followed these lines of thought, however incompletely, will deny the advantage which has accrued, and must in future accrue, from the systematic analysis of quantitative relations and symbolic operations which mainly constitute these doctrines.

Moreover, since Geometry supplies us with a highly elaborated terminology, ready to hand, and applicable to functions of not more than three dimensional variables, it is natural to extend the same phraseology to analytical investigations involving more than three

---

* On one or two occasions, I remember that questions of this sort have been brought before the Society, but without notice. Previous notice seems indispensable.

such variables. This course is so natural, indeed, that long ago Leibnitz wrote of "rectangula solida et supersolida," and no misconception could well arise.

Nevertheless, historically considered, the present adoption, in block, of geometrical phraseology into the higher algebra, is a remarkable occurrence. For you will remember, that when Algebra was painfully evolved from discrete Arithmetic, this was effected through the medium of geometrical magnitude. Thus, in the early books of the "scientific renaissance," we read of plane, solid, and sur-solid numbers. The solution of quadratic and cubic equations was originally made to depend on geometrical constructions; and Newton, in his "Universal Arithmetic," explains how expressions, "quæ ad geometriam spectant, arithmeticis tribuantur operationibus."

In later times, a change took place. Many geometrical terms fell out of use in this connection, and algebraists employed a wider phraseology. They found, as it seems to me, that the science of Quantity is larger than the science of Space, and requires terms without geometrical significance.

But now, when analysis is more that ever differentiated from its special applications, when it is even tenable that pure mathematics constitute a science of operations under laws, we are called upon to reinstate geometrical nomenclature under circumstances of great distinction and éclat.

The extension of geometrical terminology is both natural and convenient. We still retain, in Arithmetic and Algebra, such names as square, cube, fraction, the survivals of an earlier time; and other words which have almost "lost their pedigree." In fact, all language is highly figurative. Geometrical terms are borrowed for very unscientific purposes, while abstract science is deeply indebted to the vulgar tongue, and the word "Geometry" is itself fetched from a lower sphere to serve in a higher.

But I cannot regard it as otherwise than unfortunate that the unqualified transference of language, of which I speak, should take place, not as a modest matter of convenience, nor as a concession to the mental limitations which vex the dreams of those who seek a perfect language, but with pomp and blow of trumpet, as a signal victory of the empirical school.

In some cases, a praiseworthy caution has been observed. I may instance a paper by Dr. Schlaefli, "On the Integral $\int^n dx\,dy \ldots dz$" (*Quarterly Journal of Mathematics, Vol. II., pp.* 269 *et seq.*); but, as a general result, mathematicians seem to be suspected of plotting a general overthrow of our space notions.

Happily, it is not needful for me to express an opinion on the

fundamental questions in dispute. Dogmatism, in either direction, is useless, when a developed argument would be out of place.

I shall only venture to bring before you some considerations, which appear to me to establish that the progress of analysis has in no way affected the philosophical status of our notions of space and time, and that, consequently, metaphysicians need feel no alarm at what is going on behind the veil of mathematical symbolism. I think the same conclusion holds as to force and motion. You will anticipate that I do not call in question the mathematics of these new theories, or underrate the brilliant power exhibited in their development.

Let me direct your attention to the following important remarks by Riemann. After mentioning his purpose of constructing the notion of a multiply extended magnitude, he says (I quote from the late Prof. Clifford's translation):—"It will follow from this, that a multiply extended magnitude is capable of different measure-relations, and, consequently, that space is only a particular case of a triply extended magnitude. But hence it follows, as a necessary consequence, that the propositions of Geometry cannot be derived from general notions of magnitude, but that the properties which distinguish space from other conceivable triply extended magnitudes can only be deduced from experience."*

Now, since the word "magnitude" already savours of geometry, I remark that we may substitute the word "manifoldness," or simply "quantity"; and, if continuity is implied, we may say "continuum." I do not know what metaphysician or mathematician has essayed to derive the propositions of geometry from general notions of magnitude, that is to say, of "manifoldness" or "continuum."

In recent times, it has been, I believe, generally admitted, that experience furnishes the occasions on which spatial ideas are obtained and realised by intuition, abstraction or otherwise; but the phrase "deduced" ("entnommen") implies much more than this. Philosophers are at issue with the distinguished analyst on this point. One who, like Reid, maintains the instinctive perception of space, will assert that the characteristics of space, including its specialization as a triply extended magnitude of a particular kind, are involved in the perception. Another who, like Kant, holds space to be a subjective condition of externality in thought; or, like Hamilton, affirms at once an *a priori* conception and an *a posteriori* perception of space, will also affirm the implied specialization of the aggregate. Another might maintain that some, not all, of the specializing conditions are given *a priori*; and, in fact, Mr. Herbert Spencer, while classing the notions of space and its properties as *a priori* (*i.e.*, prior to individual experi-

* Riemann "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen." *Cf.* Helmholtz, *Mind*, Vol. I., p. 314.

ence), yet places the conception of space properties on a lower line. The categorical assertion of Riemann serves to determine his position in philosophic ranks, but nothing more.

The multiply extended magnitudes of which he speaks are quantitative conceptions, after all. In certain cases, we can fit them to real cognitions. They are not, in that case, the cognitions, but an expression of certain quantitative relations to which the cognitions conform. We can, indeed, do wonders by a definition; for, as Pascal says, there is nothing so free as definitions. In this wise, space may be defined as a multiply extended magnitude, having certain specified measure relations; but then we give one name to things which may be utterly heterogeneous in all but what the definition marks. Are we to consider Time as a one-dimensioned space, because it is a once-extended magnitude, having measure relations similar to those of linear space? If so, why do we not hear of elliptic and parabolic time, and time of n dimensions? Simply, I think, because such problematical cognitions are absent from our minds.

An infinite variety of spaces of two dimensions may be imagined, because we can conceive them as boundaries of space of three dimensions. We have them, finite or infinite, of constant or variable curvatures. But we cannot form any notion whatever of two-fold or one-fold space in four dimensions, because we have no cognitions to which they fit. It is admitted, on all hands, that we can form no conception whatever of a fourth geometrical dimension. I do not suppose it will be seriously asserted that we have any cognition, or conception, or notion of space of three dimensions, and constant finite curvature in the pseudo-sense.

> "These are blind fancies, reason cannot know
> What sense can neither feel nor thought conceive."

What may be the case with other possible intelligences, who can tell? We are but blind guides when we legislate for Saturn; but, if realities exist which may be adequately interpreted in these quantitative laws by means of a triply-extended magnitude of constant quasi-curvature, there is no reason to affirm that those cognitions are spatial in their character. How can we legitimately assume this, when we ourselves have different notions forming one-fold aggregates with similar measure relations, notions so different, yet so wonderfully connected, as time and linear space?

These aggregates are mathematical and subjective. As mathematically conceivable, we attach no externality to them. It is not unusual to suppose a race of beings existing in, or at least only conscious of, a two-dimensioned space; but it is equally legitimate to suppose a race rational and self-conscious, without space notions at all, but still having notions of quantity and measurement. Well, such

a race might work out the theory of a triply extended magnitude, or doubly extended magnitude, with measure relations, and yet have no notion of space of any kind.

I say, then, that these aggregates do not affect the question as to the possibility of other spaces, a question argued long before they attracted special attention, and which will be argued in future in much the same way. Mathematics are neutral in Philosophy.

The conception of space is admittedly vague and inadequate, until sustained reflection is brought to bear upon it. A savage, whose calculating faculty was limited to counting four on his fingers, and holding up his thumb to signify multitude, would hardly be expected to realise space as a triply extended magnitude of any description. Probably he would believe that colour may be solid, because a nut is white all through. This being so, it cannot be taken for granted that the most cultured person adequately conceives space, as it confronts the human mind, much less as it confronts minds of a higher order, or as it is in itself. Hence, it is quite to be expected that we should be able from time to time to analyse our spatial notions with greater precision. In this way, the researches of Lobatchewsky are extremely valuable, especially with regard to the theory of parallels. But it is right to remember that the possibility of geometries proceeding from postulates different from those of Euclid, or their equivalents, has long been recognised by philosophers. Berkeley led the way, by distinguishing between visible and tangible magnitude, which, indeed, he regarded as distinct in nature, and only related as sign to thing signified.

Reid's section on the "Geometry of Visibles,"* with his entertaining account of the "Idomenians," who only perceived visible extension, is interesting, because he gives a specimen of the Geometry,—recognising a straight line as intersecting another straight line in two points, the sum of the angles of a rectilinear triangle as being greater than two right angles, and the sum of space as being finite. The theory is, of course, reducible to a case of projection on a sphere, but is peculiar in that the two-dimensioned space is not recognised as existing on a surface.

The example suffices, if any were required, to show that our plane geometry is not the only consistent one. I do not see that, philosophically speaking, the parallel-theory of Lobatchewsky, and his "Géométrie Imaginaire," alter our position as to the space idea, except in so far as the elaboration of his work is calculated to impress the mind more forcibly.

In the mathematical discussions, it is sometimes difficult to tell

* "Reid's Collected Writings," Hamilton's Edition, Vol. I., p. 147.

whether space is regarded in its subjective or its objective aspect.* It may be observed, that when mathematicians argue the subject without their symbols, the argument is of the old sort. It is affirmed, that since experience, or rather the objects of experience, may really vary within the limits of observational error, without altering the consequent perception, the ultimate conceptions may be inconsistent with underlying facts. Obviously, experiences which differ imperceptibly will furnish the same occasions as far as the recognised physical element is concerned. It is, in a certain sense, conceivable, or may be stated hypothetically, that nature cheats us after all. If we go behind our faculties, and allege deception there, I know of no reply. Our discussions should, surely, have reference to our mental certitudes; and if it be true—I do not say it is—that our conception of space, reflected on without prejudice, is of space as infinite, for instance, it is outside all argument, mathematical or otherwise, to affirm that the conviction is illusive, since, by supposition, we can have no other. Our first object should be, to obtain correct estimates of what our notions are; our next, to account for them if we can. We must beware of cutting down cognitions to suit a pet theory of the cosmos. It is hard to get rid of the idols of the den.

Modern investigations have incidentally rehabilitated Euclid, with respect to his theory of parallels. For, it appears, he assumed neither more nor less than was necessary under the circumstances.† The Greek Geometers did not begin by describing and unfolding the idea of space. Its existence is a suppressed postulate. Euclid could not start with ideal homogeneity, continuity, and flatness. He stated properties which implied these. Prof. Clifford shows this in a wonderfully clear manner.

The definition of a straight line given among Euclid's definitions is, I believe, attributed to some editor who noticed the absence of a definition. It is too vague to be reasoned from, and is, in fact, superseded by the postulates. But you will observe that it is really an attempt to express the flatness all round and along a straight line.‡ Practically, we cannot even now derive the "Elements" from a general idea of homaloidal space.

I have detained you much too long with this order of ideas. As

---

* Sometimes space is treated like a physical object. Thus, Prof. Clifford says,—"The same faculty which tells you that space is continuous, tells you that this water is continuous, and that the motion perceived in a wheel of life is continuous."

† I do not mean to imply that Euclid's postulates are absolutely the best in form.

‡ Kant considers the proposition—straight line between two points = shortest distance—to be synthetical, because straightness is a quality not referring to quantity. There is something in this requiring explanation.

mathematicians, we like to tell or to hear some new thing. Philosophers may excusably hesitate, when the cry is "new lamps for old." Fortunately, we are all agreed that the old Geometry is accurate enough for all practical purposes. Lobatchewsky calculated that, at the time he wrote, in a right-angled isosceles triangle, whose sides equalled in length the distance from the earth to the sun, the sum of the angles could not differ from two right angles by more than 0″.0003 of a sexagesimal second. Euclid could not have had that satisfaction. Probably he never measured the angles of a triangle whose sides had the dimension of one foot. We may be sure he and his contemporaries could not measure a parasang without grievous error. He has, therefore, additional claims to our sympathy and admiration, since his ideal determinations bear the latest tests of science.

Note on Quartic Curves in Space. By William Spottiswoode, P.R.S.

[*Read Nov. 9th*, 1882.]

If we represent $ax+\beta y+\gamma z+\delta t$ by $ax+\ldots$, and put

$$\left.\begin{aligned} u &= ax+\ldots, & w &= a_2x+\ldots \\ v &= a_1x+\ldots, & s &= a_3x+\ldots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

then any quartic curves formed by the complete intersection of two quadric surfaces, viz., a quartic curve of the species (2, 2) of Cayley, may be represented by the equations

$$uy-vx=0, \quad wt-sz=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

or, if λ, μ be two indeterminate quantities, the equations (2) may be replaced by the following system:

$$\left.\begin{aligned} u &= \lambda x, & w &= \mu z \\ v &= \lambda y, & s &= \mu t \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From these we may eliminate x, y, z, t, and obtain the resultant

$$\begin{vmatrix} a-\lambda & \beta & \gamma & \delta \\ a_1 & \beta_1-\lambda & \gamma_1 & \delta_1 \\ a_2 & \beta_2 & \gamma_2-\mu & \delta_2 \\ a_3 & \beta_3 & \gamma_3 & \delta_3-\mu \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

by means of which μ may be determined in terms of λ. This having been done, we may, from any three of the equations (3), obtain values of the ratios $x:y:z:t$ in terms (not, however, generally rational)

of one parameter λ. And to every value of λ there will correspond a point, or points, real or imaginary, on the curve.

A solution of the equation (3), in a symbolical form at least, may be thus obtained. If we write,

$$(0) = \alpha,\ (1) = \beta_1,\ (2) = \gamma_2,\ (3) = \delta_3 \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$(0,1) = \alpha,\ \beta,\quad (0,2) = \alpha,\ \gamma,\ \ldots\ (2,3) = \gamma_2,\ \delta_2,\ \ldots$$
$$\alpha_1,\ \beta_1 \qquad\qquad \alpha_2,\ \gamma_2 \qquad\qquad \gamma_3,\ \delta_3,$$

$$(0,1,2) = \alpha,\ \beta,\ \gamma,\ \ldots\ ;$$
$$\alpha_1,\ \beta_1,\ \gamma_1$$
$$\alpha_2,\ \beta_2,\ \gamma_2$$

and if it be understood that the symbols 0, 1, 2, 3 are to be combined in the following way,

$$(0)\,.\,(1) = (0,1),\quad (0)\,.\,(1)\,.\,(2) = (0,1,2)\ldots\ldots\ldots\ldots(6),$$

it will be found that (4) may be written thus

$$(\lambda-0)\,(\lambda-1)\,(\mu-2)\,(\mu-3) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(7)\ ;$$

and further, if we put $(\lambda-0)\,(\lambda-1) = \Lambda$,

the equation (7) may be developed in terms of μ in the following form

$$[\Lambda]\,\mu^2 - [\Lambda\,(2+3)]\,\mu + [\Lambda\,(23)] = 0 \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

in which the terms within brackets may be treated as ordinary algebraical quantities. The solution of (8) will then be

$$2\,[\Lambda]\,\mu = [\Lambda\,(2+3)] \pm \surd\{[\Lambda\,(2+3)]^2 - 4\,[\Lambda]\,[\Lambda\,(2,3)]\}\ \ldots(9).$$

The terms in this expression, when fully developed, are

$$\begin{aligned}
\Lambda\,(2+3) &= \lambda^2\,(2+3)\\
&\quad - \lambda\,\{(0,2)+(0,3)+(1,2)+(1,3)\}\\
&\quad + (0,1,2)+(0,1,3)\\
[\Lambda\,(2+3)]^2 &= \lambda^4\,(2+3)^2\\
&\quad - 2\lambda^3\,(2+3)\,\{(0,2)+(0,3)+(1,2)+(1,3)\}\\
&\quad + \lambda^2\,\{(0,2)+(0,3)+(1,2)+(1,3)]^2\\
&\qquad\qquad + 2\,(2+3)\,[(0,1,2)+(0,1,3)]\}\\
&\quad - 2\lambda\,\{(0,2)+(0,3)+(1,2)+(1,3)\}\{(0,1,2)+(0,1,3)\}\\
&\quad + \{(0,1,2)+(0,1,3)\}^2\\
-4\,[\Lambda]\,[\Lambda\,(2,3)] &= -4\,\{\lambda^2-(0+1)\,\lambda+(0,1)\}\\
&\qquad \times\{\lambda^2\,(2,3)-\lambda\,[(0,2,3)+(1,2,3)]+(0,1,2,3)\}\\
&= -4\lambda^4\,(2,3)\\
&\quad + 4\lambda^3\,\{(0,2,3)+(1,2,3)+(0+1)\,(2,3)\}\\
&\quad - 4\lambda^2\,\{(0,1,2,3)+(0,1)\,(2,3)\\
&\qquad\qquad + (0+1)\,[(0,2,3)+(1,2,3)]\}\\
&\quad + 4\lambda\,\{(0+1)(0,1,2,3)+(0,1)[(0,2,3)+(1,2,3)]\}\\
&\quad - 4\quad (0,1)\,(0,1,2,3).
\end{aligned}$$

Hence the discriminant under the sign $\surd$ in (9)

$$
\begin{aligned}
= \quad & \lambda^4\{(2+3)^2-4(2,3)\} \\
& +2\lambda^3\{2(0,2,3)+2(1,2,3)+2(0+1)(2,3) \\
& \qquad -(2+3)[(0,2)+(0,3)+(1,2)+(1,3)]\} \\
& +\lambda^2\{[(0,2)+(0,3)+(1,2)+(1,3)]^2+2(2+3)[(0,1,2)+(0,1,3)] \\
& \qquad -4(0+1)[(0,2,3)+(1,2,3)]-4(0,1,2,3)\} \\
& +2\lambda\{2(0+1)(0,1,2,3)+2(0,1)[(0,2,3)+(1,2,3)] \\
& \qquad -[(0,2)+(0,3)+(1,2)+(1,3)][(0,1,2)+(0,1,3)]\} \\
& +\{[(0,1,2)+(0,1,3)]^2-4(0,1)(0,1,2,3)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(10).
\end{aligned}
$$

If we then take for the determination of the ratios $x : y : z : t$, the first two and the fourth equations of the system (3), we shall find

$$x : y : z : t = \begin{vmatrix} \alpha-\lambda, & \beta, & \gamma, & \delta \\ \alpha_1, & \beta_1-\lambda, & \gamma_1, & \delta_1 \\ \alpha_3, & \beta_3, & \gamma_3, & \delta_3-\mu \end{vmatrix},$$

and writing $\begin{vmatrix} \alpha-\lambda, & \beta, & \gamma, & \delta \\ \alpha_1, & \beta_1-\lambda, & \gamma_1, & \delta_1 \\ \alpha_3, & \beta_3, & \gamma_3, & \delta_3 \end{vmatrix} = A, B, C, D,$

and $\begin{vmatrix} \alpha-\lambda, & \beta, & \gamma \\ \alpha_1, & \beta_1-\lambda, & \gamma_1 \end{vmatrix} = a, b, c,$

we shall find

$$x : y : z : t = (A-a\mu) : (B-b\mu) : (C-c\mu) : D \ldots\ldots\ldots(11).$$

But, according to equation (8), μ is of the form $p \pm \surd q$; hence (11) may be written also in the following form

$$x \quad y : z : t = (A-ap \pm a\surd q) : (B-bp \pm b\surd q) : (C-cp \pm c\surd q) : D \ldots(12);$$

and consequently, for every value of λ, there are in general two values for each of the coordinates $x : y : z : t$; in other words, each value of λ in general determines two points on the curve. In fact, to each value of λ there correspond two values of μ, say μ_1, μ_2; and then to λ, μ_1 there corresponds one point, and to λ, μ_2 one point; two in all.

If the discriminant has a pair of equal roots, then, as Prof. Cayley suggests, its square root will be of the form

$$\begin{aligned} & (\lambda-\alpha)\surd(P\lambda^2+2Q\lambda+R), \\ = & (\lambda-\alpha)\surd\{(P\lambda+Q)^2-(Q^2-PR)\}, \end{aligned}$$

and, assuming $P\lambda+Q = \dfrac{\theta^2+1}{\theta^2-1}\surd(P^2-QR)$,

the foregoing expression becomes

$$(\lambda-\alpha)\frac{(\theta^2+1)}{2\theta\surd P}\surd(P^2-QR);$$

that is, λ and μ may be expressed rationally in terms of θ, and consequently the curve is unicursal.

If the discriminant either has two pairs of equal roots, or vanishes, it is clear that μ can be expressed as the ratio of two rational quadratic functions of λ, and that the curve is unicursal; but I am not at present able to make further statement of the peculiarities of these cases.

On the Explicit Integration of certain Differential Resolvents.

By Sir James Cockle, M.A., F.R.S.

[*Read Nov. 9th*, 1882.]

1. Let y be any root of

$$y^n - xy + x = 0 \quad \ldots\ldots(1),$$

and let z be any root of $xz^{n-1} - n^3 z + n^3 = 0$ $\ldots\ldots(2)$.

2. Then, if $n = 3$ or 4, we can so determine two numbers λ and μ as that

$$x^{-\lambda} y \text{ and } z^{\mu}$$

shall satisfy the same linear differential equation of the order $n-1$.

3. Put

$$xy^n - y^a + 1 = 0 \quad \ldots\ldots(3),$$

then, y being any root of (3), y^m satisfies the n-ordinal*

$$[D]^a \left[\frac{n-a}{a} D + \frac{m}{a}\right]^{n-a} y^m - \left[\frac{n}{a} D + \frac{m}{a} - 1\right]^n x^a y^m = 0 \quad \ldots(4),$$

for all values of m and for all integral values of n and a, provided that $a < n$.

4. In (3) and (4), substitute x^{-1} for x and 1 for a. Then (3) re-

---

* I have elsewhere (*Trans. Royal Society, Victoria*, Vol. vii., pp. 192, 193; 1866) shown this. I do not repeat the proof here because means of verifying the result have already been printed in the *Proceedings*. And if, at pp. 216 and 218 of Mr. Harley's Addendum to Mr. Rawson's paper (*Proc. Lond. Math. Soc.*, Vol. ix., 1878), we make the substitution

$$\begin{pmatrix} y^{-1}, & n-a, & -1, & 1, & n, & 1, & -m \\ y, & r, & b, & a, & m, & c, & n \end{pmatrix},$$

wherein the lower line refers to Mr. Harley's formulæ (i.) and (x.), we get (3) and (4) of the text. The results may moreover be transformed into, and confirmed by, a generalization of a theorem of Boole's, simultaneously and independently arrived at by Mr. Harley in England, and by me in Queensland (see *Report of British Association*, Meeting of 1866, pp. 2, 3 of "Notices and Abstracts").

duces to (1), and (4) becomes

$$-D\,[m-(n-1)\,D]^{n-1}y^m-[m-1-nD]^n\,x^{-1}y^m=0 \quad \ldots\ldots(5),$$

which is satisfied by making $y =$ any root of (1).

5. Replacing y, n, a, and x by z, $n-1$, 1, and $\frac{x}{n^3}$ respectively, (3) becomes (2), and, writing μ instead of m, we get, in place of (4),

$$n^3D\,[(n-2)\,D+\mu]^{n-2}\,z^\mu-[(n-1)\,D+\mu-1]^{n-1}\,xz^\mu=0 \quad \ldots\ldots(6),$$

n being an integer >1.

6. Let $m=1$, then (5) reduces to

$$[1-(n-1)\,D]^{n-1}\,y-n\,[-nD-1]^{n-1}\,x^{-1}y=0,$$

no arbitrary constant being introduced after the integration, because $\Sigma y=0$ when $n=3$ or 4.

7. This last $(n-1)$-ordinal is, by the algorithm of factorials, equivalent to $[(n-1)\,D+n-3]^{n-1}\,y-n\,[nD+n-1]^{n-1}\,x^{-1}\,y=0$.

8. Multiply this into $x^{1-\lambda}$, transfer the factor from left to right, and transpose the terms of the result; we get

$$n\,[nD+n\lambda-1]^{n-1}(x^{-\lambda}y)-[(n-1)\,D+(n-1)\,\lambda-2]^{n-1}x\,(x^{-\lambda}y)=0\ldots(7).$$

9. Hence (6) and (7) will, as differential equations, be identical, provided that

$$\mu=(n-1)\,\lambda-1 \quad \ldots\ldots(8),$$

$$n^3D\,[(n-2)\,D+\mu]^{n-2}=n\,[nD+n\lambda-1]^{n-1} \quad \ldots\ldots(9).$$

10. But (8) and (9) are satisfied, independently of D,* by

$$n=3, \quad \lambda=\tfrac{1}{3}, \quad \mu=-\tfrac{1}{3},$$
$$n=3, \quad \lambda=\tfrac{2}{3}, \quad \mu=\tfrac{1}{3},$$
$$n=4, \quad \lambda=\tfrac{1}{2}, \quad \mu=\tfrac{1}{2}.$$

---

* When $n=3$ or 4, then $(n-2)^{n-2}=n^{n-3}$, and (9) is of the degree $n-2$ in D. Hence (9) will be satisfied identically, if, treating D as a symbol of quantity, we can find more than $n-2$ values of D which shall satisfy (9). Let $D=0$, $\frac{\mu}{2-n}$; then (9) takes the forms $0=n\,[n\lambda-1]^{n-1}$ and $0=\left[\frac{2-n\lambda}{n-2}\right]^{n-1}$, both of which equations are satisfied by $\lambda=\frac{1}{n}$ or $\frac{2}{n}$, when $n=3$; and by $\lambda=\frac{2}{n}$, when $n=4$. If $n=4$, we require a third equation in order to establish the identity. Put $D=\frac{1}{n}-\lambda$, and (9) becomes $n^3\left(\frac{1}{n}-\lambda\right)\left[\lambda-\frac{2}{n}\right]^{n-2}=0$. Therefore $\lambda=\frac{2}{n}=\frac{1}{2}$ is an appropriate value; and there is no other. Other convenient assumptions, for instance, $D=\frac{1}{n}$ or $D=-\lambda$, which severally reduce (9) to

$$n\left[(n-1)\lambda-\frac{2}{n}\right]^{n-2}=[n\lambda]^{n-1} \text{ and } [-1]^{n-1}=-n^2\,[\lambda]^{n-1},$$

confirm the foregoing results. And an actual development of (9), in the particular cases $n=3$, 4, is thus verified.

11. Hence, for these values of n, λ, μ, the expressions $x^{-\lambda}y$ and z^{μ} satisfy the same $(n-1)$-ordinal.

12. It follows that the values of the arbitrary constants denoted by suffixed K's can always be so adjusted as to yield

$$x^{-\lambda}y_r = K_1 z_1^{\mu} + \ldots + K_{n-1} z_{n-1}^{\mu},$$

or

$$y_r = x^{\lambda}(K_1 z_1^{\mu} + \ldots + K_{n-1} z_{n-1}^{\mu}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

where y_r is any root of (1).

13. When $n=2$ the process must be slightly modified. Put $\alpha=1$, $m=1$; then (3) and (4) become

$$xy^2 - y + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

$$D(D+1)\,y - [2D]^2\,xy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

and from (12) we get successively

$$(D+1)\,y - 2(2D-1)\,xy = 2c_1$$

and the complete integral

$$y = \frac{c_1}{x} + \frac{c_2}{x}\sqrt{1-4x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

14. Take $y=\frac{1}{2}$, then $x=-2$, and by division we find that the other value of y is -1. Hence $\frac{1}{2} = -\frac{c_1}{2} + \frac{3c_2}{2}$, $-1 = -\frac{c_1}{2} - \frac{3c_2}{2}$, therefore $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$, and, with these values of c_1 and c_2, (13) is the solution of (11), the radical being taken as two-valued.

15. So long as we do not, for the purpose of defining the arbitrary constants, assume the general solution of the given equation, it is for the most part immaterial what values we assign to y. But $y=1$ must be excluded, because it would give $x=0$. And, when $n=2$, $y=2$ gives $x=\frac{1}{4}$, and, the other value of y being 2, we can only determine c_1 $(=\frac{1}{2})$. But any form of y which gives a unique value to x will suffice for the purpose. Let $y = 1 \pm \sqrt{-1}$; then $x=\frac{1}{2}$, and, as before, $c_1 = \frac{1}{2} = c_2$. Corresponding remarks apply to the cases $n=3$ or 4.

16. Take the cubic

$$y^3 - xy + x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

and, first, suppose that $\lambda = \frac{1}{3}$, $\mu = -\frac{1}{3}$. Then, by Art. 12, each of its roots is the product of $x^{\frac{1}{3}}$ into a linear and homogeneous function of the cube roots of the reciprocals of the roots of the quadratic

$$xz^2 - 27z + 27 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

17. But these roots may be taken as known, for the substitution of $27x$ for x in (15) changes (15) into (11), the general solution of which is effected in Arts. 13, 14. I add that, if in (15) we replace

z by $-\frac{27}{x^2}z$, we get the ordinary reducing quadratic, viz.,

$$z^2+xz+\frac{x^3}{27}=0.$$

18. In (14) put $y=-1$; then $x=\frac{1}{2}$, and the other two values of y are $\frac{1}{2}(1\pm\sqrt{-1})$. From (15), we get $z=27\pm15\sqrt{3}$. Hence, by (10), the expression

$$K_1(54+30\sqrt{3})^{-\frac{1}{3}}+K_2(54-30\sqrt{3})^{-\frac{1}{3}},$$

which is equivalent to

$$K_1(3+\sqrt{3})^{-1}+K_2(3-\sqrt{3})^{-1} \text{ or } \tfrac{1}{6}\{3(K_1+K_2)+(K_1-K_2)\sqrt{3}\},$$

can be made to yield the three values -1, $\frac{1}{2}(1\pm\sqrt{-1})$. And the three systems of relations

$$K_1,\ K_2=-\rho,\ -\rho^2,$$

where ρ is a cube root of 1, yield those values.

19. Secondly, let $\lambda=\frac{2}{3}$, $\mu=\frac{1}{3}$. We have substantially the same result, but with a different definition of the arbitrary constants. In fact the suppositions

$$\frac{K_1x^{\frac{1}{3}}}{z_1^{\frac{1}{3}}}+\frac{K_2x^{\frac{1}{3}}}{z_2^{\frac{1}{3}}}=C_1x^{\frac{1}{3}}z_1^{\frac{1}{3}}+C_2x^{\frac{1}{3}}z_2^{\frac{1}{3}},\quad z_1z_2=\frac{27}{x},$$

give

$$K_1z_2^{\frac{1}{3}}+K_2z_1^{\frac{1}{3}}=3\rho(C_1z_1^{\frac{1}{3}}+C_2z_2^{\frac{1}{3}}).$$

20. In (14), put $y=2$; then $x=8$ and the other two values of y are $-1\pm\sqrt{5}$. From (15), we get $z=\frac{27\pm3\sqrt{-15}}{16}$. Hence, by (10), the expression

$$C_1(108+12\sqrt{-15})^{\frac{1}{3}}+C_2(108-12\sqrt{-15})^{\frac{1}{3}},$$

which is equivalent to

$$C_1(-3+\sqrt{-15})+C_2(-3-\sqrt{-15}),$$

or

$$-3(C_1+C_2)+(C_1-C_2)\sqrt{-15},$$

can be made to yield the three values 2, $-1\pm\sqrt{5}$. And the three systems of relations

$$C_1,\ C_2=-\frac{\rho}{3},\ -\frac{\rho^2}{3},$$

yield those values. This conclusion accords with that of Art. 19, wherein we may put $\rho=1$.

21. Take the quartic $y^4-xy+x=0$.....(16).

Here $\lambda=\frac{1}{2}=\mu$, and by Art. 12 each of the roots is the product of

x^3 into a linear and homogeneous function of the square roots of the roots of the cubic $xz^3-64z+64=0$ .....(17).

22. In (16) put $y=-2$; then $x=-\frac{16}{3}$, and (16) becomes

$$\tfrac{1}{3}(y+2)[2y^3+(y-2)^3]=0 \quad \text{.....(18),}$$

so that the other three values of y are

$$2(1+\rho\sqrt[3]{2})^{-1} \text{ or } \tfrac{2}{3}(1-\rho\sqrt[3]{2}+\rho^2\sqrt[3]{4}).$$

23. From (17), we get

$$z=2\rho\sqrt[3]{2}-\rho^2\sqrt[3]{4}, \text{ and } \sqrt{z}=\frac{1}{\sqrt{3}}(2+\rho\sqrt[3]{2}-\rho^2\sqrt[3]{4}),$$

as will be seen on squaring.

24. Hence, by (10), the expression

$$\frac{4\sqrt{-1}}{3}[K_1(2+\omega\sqrt[3]{2}-\omega^2\sqrt[3]{4})+K_2(2+\omega^2\sqrt[3]{2}-\omega\sqrt[3]{4}) +K_3(2+\sqrt[3]{2}-\sqrt[3]{4})],$$

wherein ω is an unreal cube root of 1, can be made to yield the four values of y. Write i for $\sqrt{-1}$; then the four systems of relations

$4iK_1, 4iK_2, 4iK_3 = -1, -1, -1;\ -1, 1, 1;\ 1, -1, 1;\ 1, 1, -1,$

yield those values.

25. The foregoing method postulates the extraction of roots, but it does not (even in the case of a quadratic) postulate any algebraical process by which an expression can be made a perfect algebraical power. So that, in applying Tschirnhausen's process to the transformation of a general quartic into (16), we do not need to travel outside this paper.

26. In (17) change z into $-\frac{4z}{x}$; we get $z^3-4xz-x^2=0$, Descartes' cubic, which the change of z into $2z$ converts into Ferrari's cubic, and that of z into $4z$ into Euler's cubic. In (11) insert the y of (13); we get $c_1=\frac{1}{2}$, $c_2=\pm\frac{1}{2}$, which is right, the radical being taken as having either value.

On Unicursal Twisted Quartics. *By* R. A. Roberts, M.A.

[*Read Nov. 9th*, 1882.]

I propose to consider, in this paper, some properties of the unicursal twisted quartic curve, namely, the intersection of a quadric and a cubic which contains two non-intersecting generators of the quadric. In the course of my investigation, I shall almost entirely make use of

the expressions for the coordinates of a point on the curve as functions of a parameter.

Below is a brief statement of the contents :—

§§ 1, 2 treat of reduction to the canonical form.
§ 3 contains a property of the curve.
§ 4 contains the condition that four points should be complanar.
§ 5 treats of certain points on the curve.
§§ 6, 7 contain invariant conditions for degenerate forms of the curve.
§§ 8—12 treat of polygons circumscribed about the curve.
§§ 13—17 treat of circular unicursal quartics.

1. The coordinates x, y, z, u of any point on the curve being expressed in the following manner :—

$$\left.\begin{aligned} x &= a_1\lambda^4+4b_1\lambda^3\mu+6c_1\lambda^2\mu^2+4d_1\lambda\mu^3+e_1\mu^4 \equiv f_1 \\ y &= a_2\lambda^4+4b_2\lambda^3\mu+6c_2\lambda^2\mu^2+4d_2\lambda\mu^3+e_2\mu^4 \equiv f_2 \\ z &= a_3\lambda^4+4b_3\lambda^3\mu+6c_3\lambda^2\mu^2+4d_3\lambda\mu^3+e_3\mu^4 \equiv f_3 \\ u &= a_4\lambda^4+4b_4\lambda^3\mu+6c_4\lambda^2\mu^2+4d_4\lambda\mu^3+e_4\mu^4 \equiv f_4 \end{aligned}\right\} \quad \ldots\ldots\ldots(1),$$

there are easily seen to be four stationary planes, or linear functions of f_1, f_2, f_3, f_4, which are perfect fourth powers ; for, if in the equation

$$lf_1+mf_2+nf_3+pf_4 = k\,(\lambda\mu'-\lambda'\mu)^4 \quad \ldots\ldots\ldots\ldots\ldots(2),$$

we equate coefficients of λ^4, $\lambda^3\mu$, &c., and linearly eliminate l, m, n, p, k from the resulting equations, we obta

$$\begin{vmatrix} \mu'^4, & -\mu'^3\lambda', & \mu'^2\lambda'^2, & -\mu'\lambda'^3, & \lambda'^4 \\ a_1, & b_1, & c_1, & d_1, & e_1 \\ a_2, & b_2, & c_2, & d_2, & e_2 \\ a_3, & b_3, & c_3, & d_3, & e_3 \\ a_4, & b_4, & c_4, & d_4, & e_4 \end{vmatrix} = 0\,;$$

that is, we have a biquadratic to determine $\lambda' : \mu'$, which we may call the canonizant, and write

$$A\mu'^4+4B\mu'^3\lambda'+6C\mu'^2\lambda'^2+4D\mu'\lambda'^3+E\lambda'^4 = 0 \quad \ldots\ldots\ldots\ldots(3).$$

Corresponding to the four values of $\lambda' : \mu'$ there are four values of $lx+my+nz+pu$; and, referred to these planes, the curve will be determined by the equations

$$\left.\begin{aligned} x_1 &= a_1\,(\lambda-\alpha\mu)^4, \quad x_2 = a_2\,(\lambda-\beta\mu)^4 \\ x_3 &= a_3\,(\lambda-\gamma\mu)^4, \quad x_4 = a_4\,(\lambda-\delta\mu)^4 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4),$$

$\alpha, \beta, \gamma, \delta$ being the four values of $\lambda : \mu$ obtained from (3).

2. We might arrive at the equation of the canonizant biquadratic in a somewhat different form. If we equate to zero the four third

differentials of $lf_1+mf_2+nf_3+pf_4$, and eliminate l, m, n, p linearly, we obtain

$$\begin{vmatrix} a_1\lambda+b_1\mu, & b_1\lambda+c_1\mu, & c_1\lambda+d_1\mu, & d_1\lambda+e_1\mu \\ a_2\lambda+b_2\mu, & b_2\lambda+c_2\mu, & c_2\lambda+d_2\mu, & d_2\lambda+e_2\mu \\ a_3\lambda+b_3\mu, & b_3\lambda+c_3\mu, & c_3\lambda+d_3\mu, & d_3\lambda+e_3\mu \\ a_4\lambda+b_4\mu, & b_4\lambda+c_4\mu, & c_4\lambda+d_4\mu, & d_4\lambda+e_4\mu \end{vmatrix} = 0,$$

which (as at Salmon's *Higher Algebra*, Art. 169) may be seen to be identical with (3).

The invariants S and T of the canonizant are, of course, invariants of the curve, and $\frac{S^3}{T^2}$ is an absolute invariant of the curve.

3. From the equations (4) we can derive a property of the curve relating to the stationary planes. In this form it can easily be seen that the tangent to the curve is determined by the equations

$$\left.\begin{aligned} \frac{x_1}{a_1 f'(a)(\lambda-a\mu)^3} + \frac{x_2}{a_2 f'(\beta)(\lambda-\beta\mu)^3} \\ + \frac{x_3}{a_3 f'(\gamma)(\lambda-\gamma\mu)^3} + \frac{x_4}{a_4 f'(\delta)(\lambda-\delta\mu)^3} = 0 \\ \frac{ax_1}{a_1 f'(a)(\lambda-a\mu)^3} + \frac{\beta x_2}{a_2 f'(\beta)(\lambda-\beta\mu)^3} \\ + \frac{\gamma x_3}{a_3 f'(\gamma)(\lambda-\gamma\mu)^3} + \frac{\delta x_4}{a_4 f'(\delta)(\lambda-\delta\mu)^3} = 0 \end{aligned}\right\} \quad \ldots(5),$$

where $$f(t) = (t-a)(t-\beta)(t-\gamma)(t-\delta);$$

but when we are given a line

$$lx_1+mx_2+nx_3+px_4 = 0,$$
$$l'x+m'x_2+n'x_3+p'x_4 = 0,$$

the anharmonic ratio of the four points in which it meets the planes of reference is equal to $\frac{(lm'-l'm)(np'-n'p)}{(mn'-m'n)(lp'-l'p)}$

which becomes for the line (5)

$$\frac{(a-\beta)(\gamma-\delta)}{(\beta-\gamma)(a-\delta)} \text{ a constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Thus we see that the anharmonic ratio of the points in which the tangent line at any point of the curve meets the stationary planes is a constant depending on the absolute invariant $\frac{S}{T^2}$. It may be observed that this is a property of all the curves determined by the equations

$$x_1 : x_2 : x_3 : x_4 = (\lambda-a\mu)^n : (\lambda-\beta\mu)^n : (\lambda-\gamma\mu)^n : (\lambda-\delta\mu)^n.$$

4. We can now find the condition that four points on the curve should lie in a plane. If we seek the intersection of the curve with a plane, it is evident, from (4), that the equation which determines the

parameters is a linear function of the fourth powers $(\lambda-a\mu)^4$, &c., and, therefore, that the result of substituting differential symbols, viz., $\frac{d}{d\lambda}$, $-\frac{d}{d\mu}$, for μ, λ, in the canonizant, and operating on this equation must vanish. Hence, if the parameters of four complanar points be given by the equation

$$A'\mu^4+4B'\mu^3\lambda+6C'\mu^2\lambda^2+4D'\mu\lambda^3+E'\lambda^4=0 \quad \ldots\ldots\ldots\ldots(7),$$

we must have $AE'+A'E-4(BD'+B'D)+6CC'=0$ .....(8).

5. There are other canonical forms to which the curve may be reduced, as, for instance, by referring the curve to the tetrahedron formed by the points of contact of the stationary planes. If x_1 be the equation of one of the sides of this tetrahedron, and α, β, γ, δ have the same meaning as at ( ), we must have x_1 proportional to $(\lambda-\beta\mu)(\lambda-\gamma\mu)(\lambda-\delta\mu)(\lambda-\alpha'\mu)$, where α' is the parameter of the point where x_1 meets the curve again. We have then, from (8),

$$\begin{aligned}&12(\alpha\beta\gamma\delta+\alpha'\beta\gamma\delta)-3(\alpha+\beta+\gamma+\delta)(\alpha'\beta\gamma+\alpha'\gamma\delta+\alpha'\beta\delta+\beta\gamma\delta)\\&\quad-3(\alpha'+\beta+\gamma+\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\&\quad+2(\alpha\beta+\beta\gamma+\gamma\alpha+\delta\alpha+\delta\beta+\delta\gamma)(\alpha'\beta+\alpha'\gamma+\alpha'\delta+\beta\gamma+\beta\delta+\gamma\delta)=0,\end{aligned}$$

or
$$(\alpha'-\beta)(\beta-\alpha)(\gamma-\delta)^2+(\alpha'-\gamma)(\gamma-\alpha)(\beta-\delta)^2+(\alpha'-\delta)(\delta-\alpha)(\beta-\gamma)^2=0 \quad \ldots\ldots\ldots(9),$$

which determines α' in terms of α, β, γ, δ. We determine β', γ', δ' similarly, and, if U is the canonizant and H its Hessian, the equation which has these quantities for its roots is $8STH+(S^3-3T^2)U=0$. When the curve is referred to these planes, we may take

$$x_1 : x_2 : x_3 : x_4 = \frac{\lambda-\alpha'\mu}{\lambda-\alpha\mu} : \frac{\lambda-\beta'\mu}{\lambda-\beta\mu} : \frac{\lambda-\gamma'\mu}{\lambda-\gamma\mu} : \frac{\lambda-\delta'\mu}{\lambda-\delta\mu}.$$

We might also refer the curve to the tetrahedron formed by the points on the curve at which the tangents meet the curve again. To determine these points, the equation (8) must be satisfied for the coincident points $\lambda\mu$, the point $\lambda'\mu'$ where the tangent at $\lambda\mu$ meets the curve again, and an arbitrary point on the curve. We thus obtain

$$\lambda'(E\lambda^2+2D\lambda\mu+C\mu^2)+\mu'(B\lambda^2+2C\lambda\mu+D\mu^2)=0,$$
$$\lambda'(B\lambda^2+2C\lambda\mu+D\mu^2)+\mu'(A\mu^2+2B\lambda\mu+C\lambda^2)=0;$$

whence, eliminating $\lambda' : \mu'$,

$$(A\mu^2+2B\lambda\mu+C\lambda^2)(E\lambda^2+2D\lambda\mu+C\mu^2)-(B\lambda^2+2C\lambda\mu+D\mu^2)^2=0,$$

or the parameters of the required points are the roots of the Hessian of the canonizant.

6. I now proceed to find the relations which exist between the invariants S and T, when the curve has certain peculiarities. If the

curve have a node, the relation (8) must be satisfied for the parameters of the node and those of an arbitrary pair of points on the curve. Putting, then,

$$A' = \lambda_1\lambda_2\lambda_3\lambda_4, \quad -4B' = \mu_1\lambda_2\lambda_3\lambda_4 + \mu_2\lambda_1\lambda_3\lambda_4 + \mu_3\lambda_1\lambda_2\lambda_4 + \mu_4\lambda_1\lambda_2\lambda_3,$$

$$6C' = \mu_1\mu_2\lambda_3\lambda_4 + \&c., \quad -4D' = \lambda_1\mu_2\mu_3\mu_4 + \&c., \quad E' = \mu_1\mu_2\mu_3\mu_4,$$

and equating to zero the coefficients of $\lambda_3\lambda_4$, $\lambda_3\mu_4 + \lambda_4\mu_3$ $\mu_3\mu_4$, we obtain

$$A\mu_1\mu_2 + B(\lambda_1\mu_2 + \lambda_2\mu_1) + C\lambda_1\lambda_2 = 0,$$

$$B\mu_1\mu_2 + C(\lambda_1\mu_2 + \lambda_2\mu_1) + D\lambda_1\lambda_2 = 0,$$

$$C\mu_1\mu_2 + D(\lambda_1\mu_2 + \lambda_2\mu_1) + E\lambda_1\lambda_2 = 0,$$

whence

$$\begin{vmatrix} A, & B, & C \\ B, & C, & D \\ C, & D, & E \end{vmatrix} = T = 0.$$

Thus we see that $T = 0$ is the condition that the curve should have a double point. In this case the curve is the intersection of two quadrics which touch one another, and we can verify that the curve lies on two quadrics, when $T = 0$ or the roots of the canonizant are harmonically connected. Taking ∞, 0, ± 1 as the roots of the canonizant, we may write $x = \lambda^4$, $y = \mu^4$, $z = (\lambda+\mu)^4$, $u = (\lambda-\mu)^4$,

when we get

$$(2x + 2y - z - u)^2 = 144xy,$$

$$(8x + 8y - z - u)^2 = 36zu.$$

For this form it can be seen that the parameters of the node are given by $\lambda^2 + \mu^2 = 0$; but $(\lambda^2+\mu^2)^2$ is the Hessian of $\lambda\mu(\lambda^2-\mu^2)$; hence these parameters are given by the Hessian of the canonizant in this case. From (6) we see that the tangents to a unicursal quadriquadric curve with a node are divided harmonically by the stationary planes.

7. If the invariants S and T both vanish together, the parameters of the node coincide, and then the curve is the intersection of two quadrics which have stationary contact with one another. If the discriminant of the canonizant vanish, two stationary planes coincide, and give rise to a point of linear inflexion; thus we see that $S^3 = 27T^2$ is the condition that the curve should have a point of linear inflexion. If the curve have two points of linear inflexion, the corresponding conditions are obtained by expressing that the canonizant and its Hessian coincide. When S vanishes, the curve has certain peculiarities which we shall arrive at further on.

8. I now proceed to investigate a property of polygons formed by tangent lines to the curve. To find the condition that two tangents to the curve should intersect one another, we express that a plane

should touch the curve at the points of contact. We have, then, from (8), putting $A' = 1, \ -4B' = 2(\vartheta_1+\vartheta_2)$,

$$6C' = \vartheta_1^2 + \vartheta_2^2 + 4\vartheta_1\vartheta_2, \ -4D' = 2\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2), \ E' = \vartheta_1^2\vartheta_2^2,$$

$$A\vartheta_1^2\vartheta_2^2 + E + 2B\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2) + 2D(\vartheta_1+\vartheta_2) + C(\vartheta_1^2+\vartheta_2^2+4\vartheta_1\vartheta_2) = 0 \ \ldots\ldots(10),$$

where $\vartheta_1 = \frac{\mu_1}{\lambda_1}, \ \vartheta_2 = \frac{\mu_2}{\lambda_2}$.

Now this equation (10) may, by a method employed by Euler, be transformed into $\frac{d\vartheta_1}{\sqrt{H_1}} \pm \frac{d\vartheta_2}{\sqrt{H_2}} = 0$,

where $H \equiv (A\vartheta^2+2B\vartheta+C)(C\vartheta^2+2D\vartheta+E) - (B\vartheta^2+2C\vartheta+D)^2$ is the Hessian of the canonizant. Hence, integrating, we may write $u_1 \pm u_2 = a$, where u_1, u_2 are elliptic functions depending on ϑ_1, ϑ_2 by equations of the form

$$\vartheta_1 = \frac{p+q\,\mathrm{sn}^2 u_1}{p'+q'\,\mathrm{sn}^2 u_1}, \quad \vartheta_2 = \frac{p+q\,\mathrm{sn}^2 u_2}{p'+q'\,\mathrm{sn}^2 u_2}$$

and a is a constant.

If the constant a have a special value in terms of the periods $2K$ or $2iK'$ of u, there will evidently correspond a relation between the invariants S and T of the curve. By means of this result, we can arrive at a general property of polygons formed by tangents to the curve, when a certain invariant relation is satisfied. For a polygon of n sides, we have, taking the negative sign, $u_1 - u_2 = a$, $u_2 - u_3 = a$, &c., $u_n - u_{n+1} = a$, and the polygon will always close of itself if $u_{n+1} = u_1$, or, adding these equations,

$$na = \omega \ (\equiv 2mK + 2m'iK'),$$

corresponding to which relation $\frac{S^3}{T^2}$ will have a definite numerical value. Thus, when a certain invariant relation is satisfied, an infinity of skew polygons of a given number of sides can be circumscribed about the curve.

9. When $n = 3$, since three tangents cannot form a triangle, they must pass through a point. In this case we can show that S vanishes, for, if x, y, z be planes containing each pair of those tangents, and u be a stationary plane, we may take

$$x = \mu^2(\lambda+\mu)^2, \quad y = \lambda^2(\lambda+\mu)^2, \quad z = \lambda^2\mu^2, \quad u = (\lambda - k\mu)^4 \ldots(11).$$

Now, if we express that the result of operating on the canonizant

$(a, b, c, d, e)\ (\lambda, \mu)^4$ with the symbols

$$\frac{d^2}{d\lambda^2}\left(\frac{d}{d\lambda}-\frac{d}{d\mu}\right)^2, \quad \frac{d^2}{d\mu^2}\left(\frac{d}{d\lambda}-\frac{d}{d\mu}\right)^2, \quad \frac{d^4}{d\lambda^2 d\mu^2},$$

vanishes, we obtain

$$a+c-2b=0, \quad e+c-2d=0, \quad c=0,$$

whence $$S = ae-4bd+3c^2=0.$$

We can show that the locus of the points through which the tangents pass is a conic. For, if we form the invariant S for the biquadratic which gives the parameters of the points where the plane

$$\alpha x+\beta y+\gamma z+\delta u=0$$

meets the curve, we shall evidently get a quadric which bears an invariable relation to the curve. For the form (11), this surface is

$$(\alpha+\beta+\gamma+6k^2\delta)^2+12\,(1+k)\,(\alpha+k^3\beta)\,\delta=0;$$

but this evidently represents a conic passing through the point δ or xyz. We know also that, in general, there is a unique quadric containing the curve; and the plane of contact of the tangents is evidently the polar plane of their intersection with regard to this quadric. Thus we see that the plane of contact of the tangents passes through a fixed point and touches a cone of the second order.

10. If four tangents to the curve can form a skew quadrilateral, the invariant T vanishes, or the curve must be the intersection of two quadrics that touch one another. For, if we refer the curve to the tetrahedron formed by such a quadrilateral and its diagonals, we must have

$$\left.\begin{aligned} x &= (\lambda-a\mu)^2\,(\lambda-b\mu)^2, \quad y = (\lambda-c\mu)^2\,(\lambda-d\mu)^2 \\ z &= (\lambda-a\mu)^2\,(\lambda-c\mu)^2, \quad u = (\lambda-b\mu)^2\,(\lambda-d\mu)^2 \end{aligned}\right\} \quad \ldots\ldots(12).$$

Hence $xy-zu=0$, and this relation, combined with the linear relation connecting the radicals $\surd x$, &c., will give the equation of another quadric, showing that the curve lies on two quadrics. This is a property of the general quadriquadric curve; for, when four points on this curve lie on a plane, we have $u_1+u_2+u_3+u_4=0$, where u is an elliptic function, which degenerates into a logarithm or circular function if the curve has a node. When, therefore, two tangents intersect, there is $2\,(u_1+u_2)=0$ or $2mK+2m'iK'$. But, corresponding to the relations $u_1+u_2=0$, $u_2+u_3=K$, $u_3+u_4=0$, $u_4+u_1=-K$, there is evidently a quadrilateral circumscribed to the curve.

11. If a polygon of five sides be circumscribed about the curve, we may take the following expressions in the parameters for the planes containing two consecutive sides of the polygon—

$$\lambda^2\mu^2,\ \mu^2(\lambda+\mu)^2,\ (\lambda+\mu)^2(\lambda+k\mu)^2,\ (\lambda+k\mu)^2(\lambda+k'\mu)^2,\ (\lambda+k'\mu)^2\lambda^2 \ldots(13);$$

and from these we have for the canonizant $(a, b, c, d, e)(\mu\lambda)^4$,

$$c = 0, \quad a = 2b, \quad ak^2 + e = 2bk\,(k+1) + 2d\,(k+1),$$
$$ak^2k'^2 + e = 2bkk'\,(k+k') + 2d\,(k+k'), \quad e = 2dk'.$$

Hence, eliminating a and e, we get

$$d\,(k'-k-1) = bk, \quad bkk'\,(k-1) = d\,(k'-k) \quad \ldots\ldots\ldots(14);$$

therefore
$$k\,(k'-1)\,(k'-k-1) = (k-k')^2 \quad \ldots\ldots\ldots\ldots\ldots(15);$$

but
$$S = 4bd\,(k'-1), \quad T = -2bd\,(d+bk'),$$

and
$$\frac{S^3}{T^2} = \frac{16k\,(k'-1)\,(k'-k-1)}{(k-k')^2} = 16, \text{ from (15)};$$

Thus $S^3 = 16T^2$ is the condition that it should be possible to circumscribe about the curve a polygon of five sides.

12. We might find this condition otherwise thus. Suppose the canonizant to be written in one of its canonical forms, viz.,

$$\lambda^4 + \mu^4 + 6m\lambda^2\mu^2,$$

then, if $\mu : \lambda = \vartheta$, the condition that two tangents should intersect is

$$1 + \vartheta_1^2\vartheta_2^2 + m\,(\vartheta_1^2 + \vartheta_2^2 + 4\vartheta_1\vartheta_2) = 0 \ldots\ldots\ldots\ldots\ldots(16).$$

Now, suppose we have an equation

$$1 + \vartheta_1^2\vartheta_2^2 + p\,(\vartheta_1^2 + \vartheta_2^2) + 2q\vartheta_1\vartheta_2 = 0 \ldots\ldots\ldots\ldots\ldots(17),$$

corresponding to the relation between elliptic functions $u \pm v = \pm a$; then, if x, y are the roots of the equation in ϑ_2, we have

$$x + y = -\frac{2q\vartheta_1}{p + \vartheta_1^2}, \quad xy = \frac{1 + p\vartheta_1^2}{p + \vartheta_1^2},$$

whence, eliminating ϑ_1, we get

$$1 + x^2y^2 + p'\,(x^2 + y^2) + 2q'xy = 0 \quad \ldots\ldots\ldots\ldots\ldots(18),$$

where
$$p' = \frac{(1-p^2)^2}{4pq^2}, \quad q' = \frac{(1-p^2)^2}{4pq^2} - \frac{(1+p^2)}{2p}.$$

But, if u', v' are elliptic functions corresponding to x, y respectively, we have $u' = v \pm a$, $v' = \pm a - v$, whence we see that the relation $u' + v' = \pm 2a$ will correspond to (18). If we repeat this process, we get the relation corresponding to $u'' \pm v'' = \pm 4a$, and this will coincide with (17) if $5a = \omega$. Now, from (16), $p = m$, $q = 2m$; therefore
$$p'' - q'' = \frac{(1+p'^2)}{2p'} = p - q = -m, \text{ and } p' = \frac{(1-m^2)^2}{16m^3},$$

whence
$$(1 - 2m^2 + 17m^4)^2 - 256m^6\,(m^2-1) = 0,$$

which, being divided by $1 + 3m^2$, gives $(1+3m^2)^3 - 16m^2\,(1-m^2)^2 = 0$, or $S^3 - 16T^2 = 0$, as before.

The general condition, that it may be possible to circumscribe about the curve a polygon of n sides, may be most simply expressed by means of the result given by Prof. Cayley for the case of polygons circumscribed about one conic and inscribed in another (*Philosophical Magazine*, Vol. vi., p. 99). Prof. Caylev finds that, if the function $(\Delta k^3+\Theta k^2+\Theta' k+\Delta')^{\frac{1}{3}}$, where Δ, Θ, &c. are the invariants of the conics, be expanded in powers of k as follows—

$$A+Bk+Ck^2+Dk^3+\&c.,$$

then the conditions that it should be possible to have a polygon of n sides inscribed in one conic and circumscribing the other, are for $n=3, 5, 7$ respectively,

$$C=0, \quad \begin{vmatrix} C, & D \\ D, & E \end{vmatrix} = 0, \quad \begin{vmatrix} C, & D, & E \\ D, & E, & F \\ E, & F, & G \end{vmatrix} = 0 \quad \&c.,$$

and for $n=4, 6, 8,$ are

$$D=0, \quad \begin{vmatrix} D, & E \\ E, & F \end{vmatrix} = 0, \quad \begin{vmatrix} D, & E, & F \\ E, & F, & G \\ F, & G, & H \end{vmatrix} = 0, \ \&c.$$

The function to be expanded in the present case is

$$\left\{1-\frac{S^3}{4T^2}(k^3+k^2)\right\}^{\frac{1}{3}},$$

which gives the results already obtained for the cases $n=3, 4, 5,$ and will be found to give $S^3=32T^2$ for $n=6$.

13. I now proceed to investigate some properties of the circular unicursal quartic, that is, a quartic which intersects the imaginary circle at infinity four times.

Let us take $\quad x=f_1,\ y=f_2,\ z=f_3,\ u=f_4$ .....(19),

where x, y, z are rectangular planes and $u\ (=1)$ is the plane at infinity; then $f_1^2+f_2^2+f_3^2$ must have f_4 as a factor, and, therefore, we may put it equal to $f_4\phi$, where ϕ is a biquadratic in $\lambda:\mu$. If f_4 is a perfect square, the curve has double contact with the imaginary circle at infinity, and then bears some analogy to the Cartesian oval. It might be called the twisted Cartesian, but this name has been already given by Professor Sylvester to the intersection of a sphere with a parabolic cylinder.

If ϕ is a perfect square, the origin is a focus, for the point sphere $x^2+y^2+z^2=0$ has then double contact with the curve.

If the general circular curve be inverted from any point not on itself, it will become a similar curve. For the coordinates x', y', z' of

a point on the inverse curve are equal to

$$\frac{x}{x^2+y^2+z^2},\quad \frac{y}{x^2+y^2+z^2},\quad \frac{z}{x^2+y^2+z^2},$$

respectively; thus we have

$$x'=\frac{f_1}{\phi},\quad y'=\frac{f_2}{\phi},\quad z'=\frac{f_3}{\phi}.$$

If, however, the centre of inversion is on the curve, the inverse is a twisted cubic intersecting the imaginary circle at infinity twice; for f_1, f_2, f_3 have then a common factor, and since $f_1^2+f_2^2+f_3^2=f_4\phi$, this is a square factor in ϕ; whence x', &c. are proportional to cubic functions of $\lambda:\mu$. If the centre of inversion is a focus, or ϕ a perfect square, the inverse curve has double contact with the imaginary circle at infinity.

14. Since every sphere passes through four fixed points at infinity on the curve, they will intersect in four finite points. Let us take two points on the curve, the parameters corresponding to which are λ, μ; then, if S_1, S_5 be the osculating spheres at λ, μ respectively, S_2, S_4 the spheres containing the osculating circle at one point and passing through the other, and S_3 the sphere having double contact with the curve at the two points, the curve may be represented by the equations $S_1=\lambda^4$, $S_2=\lambda^3\mu$, $S_3=\lambda^2\mu^2$, $S_4=\lambda\mu^3$, $S_5=\mu^4$. In Cartesian coordinates,

$$S_i=a_i(x^2+y^2+z^2)+2f_ix+2g_iy+2h_iz+d_i,$$

and we see thus that

$$\left.\begin{aligned} S_2^2-S_1S_3=0,\quad S_3^2-S_1S_5=0,\quad S_4^2-S_3S_5=0\\ S_2S_4-S_3^2=0,\quad S_2S_5-S_3S_4=0,\quad S_1S_4-S_3S_2=0\end{aligned}\right\}\ \ldots\ldots(20)$$

represent cyclides containing the curve. An identical relation connects these six cyclides, and a linear function of five of them will represent the most general cyclide containing the curve, as a cyclide is determined by thirteen conditions, and nine are satisfied by passing through the curve. If we equate to zero the coefficient of $(x^2+y^2+z^2)^2$, we obtain the system of cubic cyclides containing the curve.

An arbitrary sphere can evidently be written in the form

$$l_1S_1+l_2S_2+l_3S_3+l_4S_4+l_5S_5=0,$$

and its intersection with the curve will then be determined by the equation $\quad l_1\lambda^4+l_2\lambda^3\mu+l_3\lambda^2\mu^2+l_4\lambda\mu^3+l_5\mu^4=0.$

Hence we can at once form the equation of the sphere passing through four given points on the curve, viz.,

$$S_1-p_1S_2+p_2S_3-p_3S_4+p_4S_5=0\ldots\ldots\ldots\ldots\ldots\ldots(21),$$

where p_1, p_2, &c. are the sum, sum of the products in pairs, &c., of the parameters $\frac{\lambda_1}{\mu_1} = \vartheta_1$, &c.

The equation of the sphere passing through the points ϑ_1, ϑ_2, ϑ_3, ϑ is evidently of the form $\Sigma + \vartheta\Sigma' = 0$, where Σ and Σ' involve ϑ_1, ϑ_2, ϑ_3 ($\Sigma = \Sigma' = 0$ is, in fact, the circle passing through ϑ_1, ϑ_2, ϑ_3). Hence, if we take three variable points ϑ_1, ϑ_2, ϑ_3, and four fixed points α, β, γ, δ, the spheres passing through the variable points and each of the fixed points are

$$\Sigma + \alpha\Sigma' = 0, \quad \Sigma + \beta\Sigma' = 0, \quad \Sigma + \gamma\Sigma' = 0, \quad \Sigma + \delta\Sigma' = 0,$$

and the anharmonic ratio of these four spheres is equal to

$$\frac{(\alpha-\beta)\,(\gamma-\delta)}{(\alpha-\delta)\,(\beta-\gamma)} = \text{a constant.}$$

By the anharmonic ratio of the spheres is understood the anharmonic ratio of their centres.

If we suppose the three variable points to coincide, we get the theorem :—The spheres described through an osculating circle and any four fixed points on the curve have a constant anharmonic ratio.

From (21) we get the equation of the sphere osculating the curve at the point ϑ, by putting

$$\vartheta_1 = \vartheta_2 = \vartheta_3 = \vartheta_4, \quad S_1 - 4\vartheta S_2 + 6\vartheta^2 S_3 - 4\vartheta^3 S_4 + \vartheta^4 S_5 = 0 \ldots (22).$$

We see at once, from this equation, that the locus of the centre of the osculating sphere is a unicursal twisted quartic ; for we have

$$x = -2\,(f_1 - 4f_2\vartheta + 6f_3\vartheta^2 - 4f_4\vartheta^3 + f_5\vartheta^4),$$
$$y = -2\,(g_1 + \&c.), \quad z = -2\,(h_1 + \&c.),$$
$$u = a_1 - 4a_2\vartheta + 6a_3\vartheta^2 - 4a_4\vartheta^3 + a_5\vartheta^4.$$

Differentiating (22), we see that the osculating circle is given by the equations

$$S_1 - 3\vartheta S_2 + 3\vartheta^2 S_3 - \vartheta^3 S_4 = 0,$$
$$S_2 - 3\vartheta S_3 + 3\vartheta^2 S_4 - \vartheta^3 S_5 = 0\,;$$

and from these we can find the locus of the centres of curvature.

The envelope of (22), that is,

$$(S_1 S_5 - 4S_2 S_4 + 3S_3^2)^3 - 27\,(S_1 S_3 S_5 + 2S_2 S_3 S_4 - S_1 S_4^2 - S_5 S_2^2 - S_3^3)^2 = 0,$$

evidently gives the locus of the osculating circles.

This surface has the given curve for a cuspidal curve, and will also have a nodal curve along which two osculating circles intersect. We find the latter curve by expressing that the equation (22) has two pairs of its roots equal. When this is the case, (22) and its Hessian coincide ; but, from (20), a linear relation connects the coefficients of the Hessian, and, therefore, the same relation connects the coefficients

of (22), which shows that the curve lies on a sphere. We can thus see that the curve is of the sixth order, and lies on a sphere.

15. The cyclide $S_1 S_5 - 4S_2 S_4 + 3S_3^2 = 0$ is evidently the locus of points through which the osculating spheres have their points of contact connected by the relation $S = 0$.

If we are given a point $P\,(S_1', S_2', S_3', S_4', S_5')$, the sphere passing through the points of contact of the osculating spheres which can be described through P is easily seen to be

$$S_1' S_5 - 4S_2' S_4 + 6S_3' S_3 - 4S_4' S_2 + S_5' S_1 = 0 \quad \ldots\ldots\ldots\ldots (23).$$

Hence, if this sphere (23) pass through a point, or the four points are complanar, the locus of P is a sphere.

If the osculating spheres at four points on the curve pass through a point, the four points are connected by a relation, which can be found by substituting $1, \frac{1}{4}p_1, \frac{1}{6}p_2, \frac{1}{4}p_3, p_4$ for S_1, S_2, S_3, S_4, S_5 respectively, in the identical relation connecting the expressions of (20).

16. For a sphere having double contact with the curve at the points ϑ_1, ϑ_2, we have, from (21), the equation

$$S_1 - 2pS_2 + (p^2 + 2q) S_3 - 2pqS_4 + q^2 S_5 = 0,$$

where $\vartheta_1 + \vartheta_2 = p$, $\vartheta_1 \vartheta_2 = q$. From this it can be seen that the locus of the centres (xyz) of these spheres is a Steiner's quartic; for we have

$$x = \frac{u_1}{u}, \quad y = \frac{u_2}{u}, \quad z = \frac{u_3}{u} \quad \ldots\ldots\ldots\ldots (24),$$

where u_1, u_2, u_3, u are functions of the second degree in p, q (see Salmon's *Surfaces*, Art. 523, Ex. 2). If r is the radius of the sphere,

there is also $$x^2 + y^2 + z^2 - r^2 = \frac{u_4}{u} \quad \ldots\ldots\ldots\ldots (25).$$

Taking the equations (19), the points of intersection of the curve and the sphere $x^2 + y^2 + z^2 - 2ax - 2\beta y - 2\gamma z + k^2 = 0$ $\ldots\ldots\ldots\ldots (26)$ are determined by the equation

$$f_4 + k^2\phi - 2\,(af_1 + \beta f_2 + \gamma f_3) = \psi\,(\lambda, \mu) = 0 \quad \ldots\ldots\ldots\ldots (27).$$

If the sphere (26) have double contact with the curve, we must have

$$\psi\,(\lambda, \mu) = (A\lambda^2 + B\lambda\mu + C\mu^2)^2,$$

and substituting in this identity the four roots of $\phi = 0$, successively, and eliminating A, B, C, we obtain a relation of the form

$$l\sqrt{X} + m\sqrt{Y} + n\sqrt{Z} + p\sqrt{U} = 0,$$

where $$X = 2\,(af_1 + \beta f_2 + \gamma f_3) - f_4, \quad Y = \&c.$$

This equation represents a Steiner's quartic, and, since $f_1^2 + f_2^2 + f_3^2 = 0$

for a root of ϕ, the four planes X, Y, Z, U touch the imaginary circle at infinity.

17. If we put $r = 0$ in (25), it can be seen from (24) that the locus of the centres of spheres of no radius which have double contact with the curve, is a curve of the eighth order meeting the imaginary circle at infinity eight times. We see thus that the focal curve is of the eighth order, and lies on a Steiner's quartic. When the curve has a double point, in which case it is the intersection of a sphere and a quadric touching one another, we know that there are two focal curves, each of the fourth order, lying on spheres; and this agrees with the result obtained above.

Putting $k^2 = x^2+y^2+z^2$ in (27), and giving $\lambda : \mu$ a special value $\lambda_1 : \mu_1$, we have $\sqrt{\phi_1}\,\rho_1 = A_1\lambda_1^2 + B\lambda_1\mu_1 + C\mu_1^2$ .....(28),

where ρ_1 denotes the distance of any point on the focal curve from the point $\lambda_1 : \mu_1$. Hence we see that a linear homogeneous relation will connect the distances of any point on the focal curve from four fixed points on the given curve.

Again, from (27), suppose four foci to be given, then A_1, B_1, C_1, &c. are constants, and we see that there is a linear homogeneous relation connecting the distances of any point on the curve from four fixed foci. There is also, evidently, a homogeneous relation of the second degree, connecting the distances of any point on the curve from three fixed foci.

If the curve have double contact with the imaginary circle at infinity, ϕ is a perfect square $= (a\vartheta^2+b\vartheta+c)^2$, say, putting $\lambda : \mu = \vartheta$. We have, then, for the distance of any point of this curve from a focus $\rho_1 = \dfrac{A_1\vartheta^2+B_1\vartheta+C_1}{a\vartheta^2+b\vartheta+c}$; and, therefore, taking three foci, there is a relation of the form $l\rho_1+m\rho_2+n\rho_3 = d$. For this curve it can be shown that the focal curve is of the seventh degree, and lies on a cubic surface with a double line. In this case, in fact, ϕ is a perfect square, and the differentials of (27) will also be satisfied by its roots ϑ_1, ϑ_2; thus, if x, y, z, u are four planes, we may take

$$x = (A\vartheta_1^2+2B\vartheta_1+C)(A\vartheta_1+B), \quad y = (A\vartheta_1^2+2B\vartheta_1+C)(C+B\vartheta_1),$$

$$z = (A\vartheta_2^2+2B\vartheta_2+C)(A\vartheta_2+B), \quad u = (A\vartheta_2^2+2B\vartheta_2+C)(C+B\vartheta_2),$$

whence $(u+\vartheta_1 z)^2(y+\vartheta_1 x) = (y+\vartheta_2 x)^2(u+\vartheta_2 z)$.

On In-and-Circumscribed Polyhedra. *By* Prof. A. R. Forsyth.

[*Read Nov. 9th*, 1882.]

This paper is intended to solve for quadric surfaces a problem which corresponds to the porism of polygons in-and-circumscribed to two conics, the case considered here being that of polyhedra which are circumscribed to one quadric and have pairs of opposite edges lying along the surface of another. It will be seen that the method here adopted is the natural generalisation of that used by Prof. Cayley in the discussion of the porism of polygons in-and-circumscribed to two conics;* the limited case when the conics are both circles had been previously discussed by Fuss, Steiner, Jacobi, Richelot, and Minding.†

Consider two quadrics and refer them to their common self-conjugate tetrahedron, then their equations may be taken as

$$U \equiv ax^2+by^2+cz^2+dw^2 = 0,$$

$$V \equiv x^2+y^2+z^2+w^2 = 0,$$

and any quadric through their curves of intersection will be given by

$$U+\xi V = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

in which ξ may be looked upon as a parameter defining the surface. Any plane will touch three of the surfaces, say those given by ξ, η, ζ; and η and ζ may be looked upon as parameters of the positions of points lying on the surface ξ, these being the points of contact of tangent planes to ξ. It is not difficult to prove that the equation to such a plane is

$$x(\mathrm{a}.a+\xi.a+\eta.a+\zeta)^{\frac{1}{2}}+y(\mathrm{b}.b+\xi.b+\eta.b+\zeta)^{\frac{1}{2}}$$
$$+z(\mathrm{c}.c+\xi.c+\eta.c+\zeta)^{\frac{1}{2}}+z(\mathrm{d}.d+\xi.d+\eta.d+\zeta)^{\frac{1}{2}} = 0,$$

where a, b, c, d are given by the equations,

$$\left.\begin{aligned} \mathrm{a}+\ \mathrm{b}+\ \mathrm{c}+\ \mathrm{d} &= 0 \\ a\mathrm{a}+b\mathrm{b}+c\mathrm{c}+d\mathrm{d} &= 0 \\ a^2\mathrm{a}+b^2\mathrm{b}+c^2\mathrm{c}+d^2\mathrm{d} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

---

* *Phil. Mag.*, Ser. 4, Vols. 5, 6, 7, 1853 and 1854; *Phil. Trans.*, 1861.

† For particular references, *cf.* Cayley's papers just cited.

say,
$$\mathrm{a} = -\overline{b-c}\ \overline{c-d}\ \overline{d-b},$$
$$\mathrm{b} = +\ \overline{c-d}\ \overline{d-a}\ \overline{a-c},$$
$$\mathrm{c} = -\overline{d-a}\ \overline{a-b}\ \overline{b-d},$$
$$\mathrm{d} = \ \overline{a-b}\ \overline{b-c}\ \overline{c-a},$$

or, if
$$b-c = A, \quad c-a = B, \quad a-b = C,$$
$$a-d = F, \quad b-d = G, \quad c-d = H,$$

then $\mathrm{a} = AHG, \quad \mathrm{b} = HBF, \quad \mathrm{c} = GFC, \quad \mathrm{d} = ABC,$

The parameter of the quadric $V = 0$ is ∞; let θ, ϕ be the parameters of a point on it; then the coordinates x', y', z', w' are given by

$$x' : y' : z' : w'$$
$$= (\mathrm{a}.a+\theta.a+\phi)^{\frac{1}{2}} : (\mathrm{b}.b+\theta.b+\phi)^{\frac{1}{2}} : (\mathrm{c}.c+\theta.c+\phi)^{\frac{1}{2}} : (\mathrm{d}.d+\theta.d+\phi)^{\frac{1}{2}}.$$

Suppose now that θ', ϕ' is some other point of V; the coordinates of any point on the straight line joining them may be taken as

$$\lambda\,(\mathrm{a}.a+\theta.a+\phi)^{\frac{1}{2}} + \mu\,(\mathrm{a}.a+\theta'.a+\phi'),$$
$$\lambda\,(\mathrm{b}.b+\theta.b+\phi)^{\frac{1}{2}} + \mu\,(\mathrm{b}.b+\theta'.b+\phi'),$$
$$\lambda\,(\mathrm{c}.c+\theta.c+\phi)^{\frac{1}{2}} + \mu\,(\mathrm{c}.c+\theta'.c+\phi'),$$
$$\lambda\,(\mathrm{d}.d+\theta.d+\phi)^{\frac{1}{2}} + \mu\,(\mathrm{d}.d+\theta'.d+\phi'),$$

with the relation $\lambda + \mu = 1$. If such a point lie on the quadric V for values of λ and μ other than zero or unity, then the straight line is obviously a generator. Substituting in the equation, the condition for this is

$$\mathrm{a}\,(a+\theta.a+\phi.a+\theta'.a+\phi')^{\frac{1}{2}} + \ldots + \mathrm{d}\,(d+\theta.d+\phi.d+\theta'.d+\phi')^{\frac{1}{2}} = 0.$$

To obtain a rationalised equivalent of this, we may either write, by means of equations (2),

$$(x+\theta.x+\phi.x+\theta'.x+\phi')^{\frac{1}{2}} = \alpha+\beta x+\gamma x^2,$$

for $x=a$, $x=b$, $x=c$, $x=d$, and then eliminate α, β, and γ from four relations such as

$$\theta+\phi+\theta'+\phi'+(1-\gamma^2)(a+b+c+d) = 2\beta\gamma;$$

or we may proceed as follows:—Take θ', ϕ' to be a point on the surface consecutive to θ, ϕ, so that we may write

$$\theta' = \theta+\vartheta, \quad \phi' = \phi+\chi,$$

where ϑ and χ are both small. Then

$$(a+\theta.a+\phi.a+\theta'.a+\phi')^{\frac{1}{2}} = a+\theta.a+\phi\left(1+\frac{\vartheta}{a+\theta}\right)^{\frac{1}{2}}\left(1+\frac{\chi}{a+\phi}\right)^{\frac{1}{2}}$$
$$= a+\theta.a+\phi\left[1+\tfrac{1}{2}\frac{\vartheta}{a+\theta}+\tfrac{1}{2}\frac{\chi}{a+\phi}-\tfrac{1}{8}\left(\frac{\vartheta}{a+\theta}-\frac{\chi}{a+\phi}\right)^2\right],$$

on expanding and retaining terms up to the second order inclusive;

and the equation to the generator becomes

$$\Sigma a \,.\, a+\theta \,.\, a+\phi$$
$$+\tfrac{1}{2}\vartheta\Sigma a \,.\, a+\phi+\tfrac{1}{2}\chi\Sigma a \,.\, a+\theta+\tfrac{1}{4}\vartheta\chi\,\Sigma a$$
$$-\tfrac{1}{8}\vartheta^2\Sigma a\,\frac{a+\phi}{a+\theta}-\tfrac{1}{8}\chi^2\Sigma a\,\frac{a+\theta}{a+\phi}+\text{terms of higher order}=0,$$

where Σ implies summation for the four letters a, b, c, d. Obviously, by our equations (2), the first term, the terms of the first order, and the term involving $\vartheta\chi$ all disappear; and hence, when we make ϑ, χ small, we have, as the relation connecting them,

$$\vartheta^2\Sigma a\,\frac{a+\phi}{a+\theta}+\chi^2\Sigma a\,\frac{a+\theta}{a+\phi}=0,$$

or
$$\vartheta^2\Phi\Sigma a \,.\, a+\phi \,.\, b+\theta \,.\, c+\theta \,.\, d+\theta$$
$$+\chi^2\Theta\Sigma a \,.\, a+\theta \,.\, b+\phi \,.\, c+\phi \,.\, d+\phi=0,$$

where
$$\Phi = a+\phi \,.\, b+\phi \,.\, c+\phi \,.\, d+\phi,$$
$$\Theta = a+\theta \,.\, b+\theta \,.\, c+\theta \,.\, d+\theta.$$

Let
$$\Delta_1 = abcd, \quad \Delta = ab+ac+ad+bc+bd+cd,$$
$$\Delta_2 = bcd+cda+dab+abc, \quad \Delta_3 = a+b+c+d;$$

then
$$bc+cd+db = \Delta-a\Delta_3+a^2,$$
$$ab+ac+ad = a\Delta_3-a^2,$$
$$abc+acd+adb = \Delta_2-bcd.$$

Now
$$\Sigma a \,.\, a+\phi \,.\, b+\theta \,.\, c+\theta \,.\, d+\theta$$
$$= \phi\Sigma\left[a\left\{bcd+\theta\,(bc+cd+db)+(b+c+d)\,\theta^2+\theta^3\right\}\right]$$
$$+\Sigma\left[a\left\{abcd+\theta\,(abc+acd+adb)+\theta^2\,(ab+ac+ad)+a\theta^3\right\}\right].$$

Now, in virtue of the identities just written down, and the equations giving the quantities a, the first line on the right-hand side reduces to the term $\phi\,(abcd+bcda+cdab+dabc)$,

say to
$$\phi\ \Sigma\,(abcd),$$

and the second line gives
$$-\theta\ \Sigma\,(abcd);$$

so that the equation to the generator is

$$\left[\vartheta^2\Phi\,(\phi-\theta)+\chi^2\Theta\,(\theta-\phi)\right] \,.\, \Sigma\,(abcd)=0.$$

Now $\Sigma\,(abcd)$ does not vanish, and θ and ϕ, being parameters of different quadrics, are unequal; and hence we have

$$\vartheta^2\Phi-\chi^2\Theta=0,$$

or, writing
$$\vartheta = d\theta,$$
$$\chi = d\phi,$$

this is equivalent to $$\left(\frac{d\theta}{\sqrt{\Theta}}\right)^2 - \left(\frac{d\phi}{\sqrt{\Phi}}\right)^2 = 0,$$

which is, of course, equivalent to the two

$$\frac{d\theta}{\sqrt{\Theta}} - \frac{d\phi}{\sqrt{\Phi}} = 0,$$

$$\frac{d\theta}{\sqrt{\Theta}} + \frac{d\phi}{\sqrt{\Phi}} = 0,$$

and these are the differential equations to the generators on the surface $V = 0$.* There are, as it is obvious there should be, two distinct equations corresponding to the two distinct systems.

Consider now a tetrahedron, two pairs of opposite edges of which are formed by a pair of generators from each system, and the remaining edges by lines joining the intersections of generators of different systems. Let (Fig. 1) A, B, C, D be respectively the points $\theta_1\phi_1$, $\theta_2\phi_2$, $\theta_3\phi_3$, $\theta_4\phi_4$; and suppose AB and CD belong to the first equation, and BC and DA to the second. Then writing

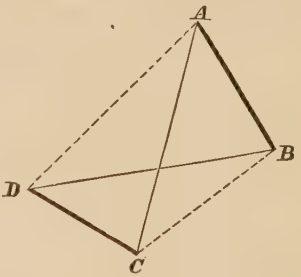


Fig. 1.

$$\int \frac{d\theta}{\sqrt{\Theta}} = u, \quad \int \frac{d\phi}{\sqrt{\Phi}} = v,$$

we have $u - v$ constant along AB and CD,

$u + v$ „ „ AD and BC,

so that

$$u_1 - v_1 = u_2 - v_2,$$

$$u_3 - v_3 = u_4 - v_4,$$

$$u_3 + v_3 = u_2 + v_2,$$

$$u_1 + v_1 = u_4 + v_4;$$

---

* See Cayley "On Geodesic Lines, in particular those of a Quadric Surface," *Proc. Lond. Math. Soc.*, t. iv., 1872, p. 199 : viz., equation of surface being

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

and expressions of coordinates in terms of parameters p and q being

$$-\beta\gamma\, x^2 = a\,.\,a+p\,.\,a+q,$$

$$-\gamma\alpha\, y^2 = b\,.\,b+p\,.\,b+q,$$

$$-\alpha\beta\, z^2 = c\,.\,c+p\,.\,c+q,$$

then differential equation of right lines is

$$\frac{dp}{\sqrt{a+p\,.\,b+p\,.\,c+p}} + \frac{dq}{\sqrt{a+q\,.\,b+q\,.\,c+q}} = 0.$$

from which we at once obtain

$$u_1+u_3=u_2+u_4 \quad \ldots\ldots\ldots\

To find what this theorem becomes when $\theta_1, \theta_2, \theta_3, \theta_4$ are all zero, so that the faces of the tetrahedron all touch the quadric $U=0$, we must expand $\sqrt{\square\theta_1}$ in powers of θ_1; then, if

$$\sqrt{\square\theta_1} = A_0 + A_1\theta_1 + \ldots + A_n\theta_1^n + \ldots,$$

the foregoing determinant becomes divisible by

$$\begin{vmatrix} 1, & \theta_1, & \theta_1^2, & \theta_1^3 \\ 1, & \theta_2, & \theta_2^2, & \theta_2^3 \\ 1, & \theta_3, & \theta_3^2, & \theta_3^3 \\ 1, & \theta_4, & \theta_4^2, & \theta_4^3 \end{vmatrix},$$

and leaves as the required condition

$$A_3 = 0.$$

We shall afterwards return to this condition.

A tetrahedron is a solid proper, *i.e.*, one which has all its solid angles contained by the same number of plane angles, and all its faces bounded by the same number of straight lines, each of these numbers in the case already considered being three. In order that such a solid may have edges lying along the generators of a quadric, it must have triangles for its faces, since there are only two generators of a quadric passing through any point on it; and therefore the only other solids which can be so described are the octahedron and the icosahedron.

We proceed to consider first the octahedron. Let the angular points be denoted by 1, 2, 3, 4, 5, 6 (Fig. 2), and let the edges 12, 23, 34, 45, 56, 61 lie along generators of the quadric V; then any point, as 1, may be denoted by ∞, θ_1, ϕ_1. Let the thick lines belong to one system, the dotted to the other; then, by joining 15, 53, 31, 42, 26, 64 by the thin lines, we have an octahedron which has three central planes 1643, 3265, 5421.

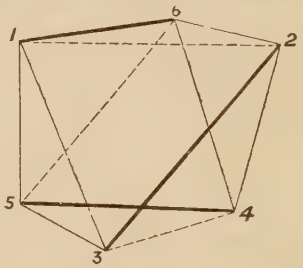


Fig. 2.

Then, adopting the same notation as in the case of the tetrahedron, we have

$$\begin{aligned} u_1 - v_1 &= u_6 - v_6 \text{ along } 16, \\ u_3 - v_3 &= u_2 - v_2 \ \ldots\ldots\ 23, \\ u_5 - v_5 &= u_4 - v_4 \ \ldots\ldots\ 45, \\ u_1 + v_1 &= u_2 + v_2 \ \ldots\ldots\ 12, \\ u_3 + v_3 &= u_4 + v_4 \ \ldots\ldots\ 34, \\ u_5 + v_5 &= u_6 + v_6 \ \ldots\ldots\ 56; \end{aligned}$$

and these equations at once give

$$u_1+u_3+u_5 = u_2+u_4+u_6,$$

$$v_1+v_3+v_5 = v_2+v_4+v_6.$$

Taking either of these equations, say the first, its algebraical equivalent is, by Abel's theorem,

$$\{1, \theta, \theta^2, \theta^3, \sqrt{\square\theta}, \theta\sqrt{\square\theta}\} = 0,$$

where the expression $\{\ \}$ denotes the determinant of six lines formed by substituting for θ the values $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$. Moreover, $\square\theta$ is the discriminant of $U+\theta V$ in the forms assumed for their equations, but, being an invariant, it may be replaced by the same value as before, viz.,

$$\Delta'+\Theta'\theta+\Phi\theta^2+\Theta\theta^3+\Delta\theta^4,$$

and U, V may have the most general forms possible. Hence we have the theorem :—

If an octahedron be described having three pairs of opposite edges lying on the surface of a quadric V, and, of its plane faces formed by these edges, five touch the respective quadrics

$$U+\theta_\mu V = 0$$

($\mu = 1, 2, 3, 4, 5$), then the sixth will touch the quadric

$$U+\theta_6 V = 0,$$

where θ_6 is given by the equation

$$\{1, \theta, \theta^2, \theta^3, \sqrt{\square\theta}, \theta\sqrt{\square\theta}\} = 0$$

(the determinant of six rows obtained by substituting for θ the values $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$), in which $\square\theta$ denotes the discriminant of

$$U+\theta V = 0.$$

To find the condition that all these six faces may touch the same quadric $U = 0$, so that $\theta_1 = \ldots\ldots = \theta_6 = 0$, we must expand $\sqrt{\square\theta}$; taking the same value as before, the determinant divides by

$$\{1, \theta, \theta^2, \theta^3, \theta^4, \theta^5\},$$

and leaves as the required condition

$$\begin{vmatrix} A_3, & A_4 \\ A_4, & A_5 \end{vmatrix} = 0,$$

or

$$A_3A_5 = A_4^2.$$

Now, consider the icosahedron which has 12 angular points and 30 edges. Since each edge joins two angular points, and through any point on the quadric V only two generators can be drawn, it follows that there will be twelve edges of the solid lying on the surface, six of them being generators of one system, and six of the other. Of the accompanying figures, Figure 3 shows these on the surface of the

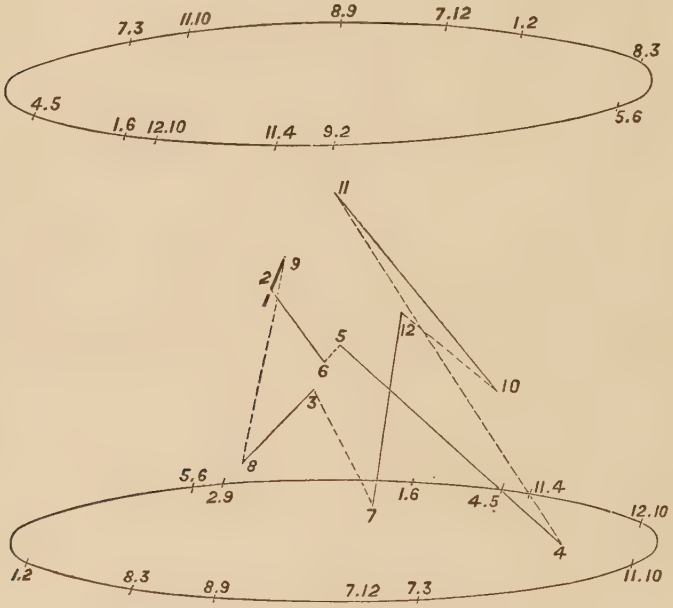


Fig. 3.

quadric (assumed a hyperboloid of one sheet), the two sections being equidistant from and parallel to the principal section, the numbers at each generator being the numbers of the angular points which lie on it; and Figure 4 is an icosahedron, more regular in form, with the angular points numbered exactly as in Figure 3, and the remaining edges formed by the thin lines joining the angular points. The dark and dotted lines indicate, as before, generators belonging to the respective systems.

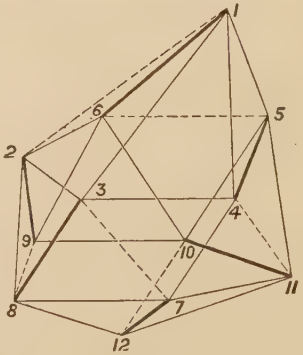


Fig. 4.

Denoting the point μ by ∞, θ_μ, ϕ_μ,

we have along

$$\begin{aligned}
1.2 \ldots\ldots & \quad u_1 + v_1 = u_2 + v_2,\\
5.6 \ldots\ldots & \quad u_5 + v_5 = u_6 + v_6,\\
8.9 \ldots\ldots & \quad u_9 + v_9 = u_8 + v_8,\\
3.7 \ldots\ldots & \quad u_3 + v_3 = u_7 + v_7,\\
10.12 \ldots\ldots & \quad u_{12} + v_{12} = u_{10} + v_{10},\\
11.4 \ldots\ldots & \quad u_{11} + v_{11} = u_4 + v_4,\\
1.6 \ldots\ldots & \quad u_1 - v_1 = u_6 - v_6,\\
2.9 \ldots\ldots & \quad u_9 - v_9 = u_2 - v_2,\\
4.5 \ldots\ldots & \quad u_5 - v_5 = u_4 - v_4,\\
3.8 \ldots\ldots & \quad u_3 - v_3 = u_8 - v_8,\\
12.7 \ldots\ldots & \quad u_{12} - v_{12} = u_7 - v_7,\\
11.10 \ldots\ldots & \quad u_{11} - v_{11} = u_{10} - v_{10};
\end{aligned}$$

from which we have

$$u_1+u_3+u_5+u_9+u_{11}+u_{12} = u_2+u_4+u_6+u_7+u_8+u_{10},$$

and the same equation in the v's. Again applying Abel's theorem, the algebraical equivalent of this is

$$\{1, \theta, \ldots, \theta^6, \sqrt{\square\theta}, \ldots, \theta^4\sqrt{\square\theta}\} = 0,$$

where the expression $\{\ \}$ denotes the determinant of 12 rows obtained by substituting for θ the values $\theta_1, \theta_2, \ldots, \theta_{12}$. Replacing, as before, the particular form of $\square\theta$ by the discriminant of $U+\theta V = 0$ when U and V are the general equations, we have the above as the condition that an icosahedron, having six pairs of opposite edges lying on the surface of the quadric $V=0$, should have its twelve faces formed by these edges respectively touching the quadrics

$$U+\theta_\mu V = 0,$$

where $\mu = 1, 2, \ldots, 12$.

When these faces all touch $U=0$, the condition becomes

$$\begin{vmatrix}
A_3, & A_4, & A_5, & A_6, & A_7\\
A_4, & A_5, & A_6, & A_7, & A_8\\
A_5, & A_6, & A_7, & A_8, & A_9\\
A_6, & A_7, & A_8, & A_9, & A_{10}\\
A_7, & A_8, & A_9, & A_{10}, & A_{11}
\end{vmatrix} = 0,$$

the determinant having been divided by

$$\{1, \theta, \theta^2, \ldots, \theta^{11}\}.$$

We have written

$$(\Delta'+\Theta'\theta+\Phi\theta^2+\Theta\theta^3+\Delta\theta^4)^{\frac{1}{2}} = A_0+A_1\theta+\ldots+A_n\theta^n+\ldots,$$

and the conditions requisite for each case have been expressed in terms of the coefficients on the right-hand side. The values of these,

as far as A_{11}, are as follow :—

$$A_0\Delta'^{-\frac{1}{2}} = 1,$$

$$A_1\Delta'^{\frac{1}{2}} = \frac{1}{2}\Theta',$$

$$A_2\Delta'^{\frac{3}{2}} = \frac{1}{2}\Phi\Delta' - \frac{1}{8}\Theta'^2,$$

$$A_3\Delta'^{\frac{5}{2}} = \frac{1}{2}\Theta\Delta'^2 - \frac{1}{4}\Theta'\Delta'\Phi + \frac{1}{16}\Theta'^3,$$

$$A_4\Delta'^{\frac{7}{2}} = \frac{1}{2}\Delta\Delta'^3 - \frac{1}{8}\Phi^2\Delta'^2 - \frac{1}{4}\Theta\Theta'\Delta'^2 + \frac{3}{16}\Theta'^2\Phi\Delta' - \frac{5}{128}\Theta'^4,$$

$$A_5\Delta'^{\frac{9}{2}} = -\frac{1}{4}\Theta'\Delta\Delta'^3 - \frac{1}{4}\Phi\Theta\Delta'^3 + \frac{3}{16}\Theta'^2\Theta\Delta'^2 + \frac{3}{16}\Phi^2\Theta'\Delta'^2 - \frac{5}{32}\Theta'^3\Phi\Delta' + \frac{7}{256}\Theta'^5,$$

$$A_6\Delta'^{\frac{11}{2}} = -\frac{1}{8}\Theta^2\Delta'^4 - \frac{1}{4}\Delta\Phi\Delta'^4 + \frac{1}{16}\Phi^3\Delta'^3 + \frac{3}{16}\Theta'^2\Delta\Delta'^3 + \frac{3}{8}\Theta\Theta'\Phi\Delta'^3 - \frac{5}{32}\Theta\Theta'^3\Delta'^2 - \frac{15}{64}\Theta'^2\Phi^2\Delta'^2 + \frac{35}{256}\Theta'^4\Phi\Delta' - \frac{21}{1024}\Theta'^6,$$

$$A_7\Delta'^{\frac{13}{2}} = -\frac{1}{4}\Theta\Delta\Delta'^5 + \frac{3}{16}\Phi^2\Theta\Delta'^4 + \frac{3}{16}\Theta^2\Theta'\Delta'^4 + \frac{3}{8}\Phi\Theta'\Delta\Delta'^4 - \frac{5}{32}\Theta'^3\Delta\Delta'^3 - \frac{5}{32}\Theta'\Phi^3\Delta'^3 - \frac{15}{32}\Theta'^2\Phi\Theta\Delta'^3 + \frac{35}{256}\Theta'^4\Theta\Delta'^2 + \frac{35}{128}\Theta'^3\Phi^2\Delta'^2 - \frac{63}{512}\Theta'^5\Phi\Delta' + \frac{33}{2048}\Theta'^7,$$

$$A_8\Delta'^{\frac{15}{2}} = -\frac{1}{8}\Delta^2\Delta'^6 + \frac{3}{16}\Phi^2\Delta\Delta'^5 + \frac{3}{16}\Theta^2\Phi\Delta'^5 + \frac{3}{8}\Theta\Theta'\Delta\Delta'^5 - \frac{15}{32}\Theta'^2\Phi\Delta\Delta'^4 - \frac{15}{64}\Theta^2\Theta'^2\Delta'^4 - \frac{15}{32}\Theta\Theta'\Phi^2\Delta'^4 - \frac{5}{128}\Phi^4\Delta'^4 + \frac{35}{256}\Theta'^4\Delta\Delta'^3 + \frac{35}{64}\Theta'^3\Phi\Theta\Delta'^3 + \frac{35}{128}\Theta'^2\Phi^3\Delta'^3 - \frac{63}{512}\Theta'^5\Theta\Delta'^2 - \frac{315}{1024}\Theta'^4\Phi^2\Delta'^2 + \frac{231}{2048}\Theta'^6\Phi\Delta' - \frac{429}{32768}\Theta'^8,$$

$$A_9\Delta'^{\frac{17}{2}} = \frac{1}{16}\Theta^3\Delta'^6 + \frac{3}{16}\Delta^2\Theta'\Delta'^6 + \frac{3}{8}\Phi\Theta\Delta\Delta'^6 - \frac{15}{32}\Theta'^2\Theta\Delta\Delta'^5 - \frac{15}{32}\Theta'\Phi^2\Delta\Delta'^5 - \frac{15}{32}\Theta'\Phi\Theta^2\Delta'^5 - \frac{5}{32}\Phi^3\Theta\Delta'^5 + \frac{35}{228}\Theta'^3\Theta^2\Delta'^4 + \frac{35}{64}\Theta'^3\Phi\Delta\Delta'^4 + \frac{105}{128}\Theta'^2\Phi^2\Theta\Delta'^4 + \frac{35}{256}\Theta'\Phi^4\Delta'^4 - \frac{63}{512}\Theta'^5\Delta\Delta'^3 - \frac{315}{512}\Theta'^4\Phi\Theta\Delta'^3 - \frac{105}{256}\Theta'^3\Phi^3\Delta'^3 + \frac{231}{2048}\Theta'^6\Theta\Delta'^2 + \frac{693}{2048}\Theta'^5\Phi^2\Delta'^2 - \frac{429}{4096}\Theta'^7\Phi\Delta' - \frac{715}{65536}\Theta'^9,$$

$$
\begin{aligned}
A_{10}\Delta'^{\frac{1}{2}} = {} & \frac{3}{16}\Theta^2\Delta\Delta'^7 + \frac{3}{16}\Phi\Delta^2\Delta'^7 - \frac{15}{64}\Theta'^2\Delta^2\Delta'^6 - \frac{15}{64}\Theta'\Phi\Theta\Delta\Delta'^6 \\
& - \frac{5}{32}\Theta'\Theta^3\Delta'^6 - \frac{5}{32}\Phi^3\Delta\Delta'^6 - \frac{15}{64}\Phi^2\Theta^2\Delta'^6 + \frac{35}{64}\Theta'^3\Theta\Delta\Delta'^5 \\
& + \frac{105}{128}\Theta'^2\Phi^2\Delta\Delta'^5 + \frac{105}{128}\Theta^2\Theta'^2\Phi\Delta'^5 + \frac{35}{64}\Theta'\Phi^3\Theta\Delta'^5 + \frac{7}{256}\Phi^5\Delta'^5 \\
& - \frac{315}{1024}\Theta'^4\Theta^2\Delta'^4 - \frac{315}{512}\Theta'^4\Phi\Delta\Delta'^4 - \frac{315}{256}\Theta'^3\Phi^2\Theta\Delta'^4 \\
& - \frac{315}{1024}\Theta'^2\Phi^4\Delta'^4 + \frac{231}{2048}\Theta'^6\Delta\Delta'^3 + \frac{693}{1024}\Theta'^5\Theta\Phi\Delta'^3 \\
& + \frac{1155}{2048}\Theta'^4\Phi^3\Delta'^3 - \frac{429}{4096}\Theta^7\Theta\Delta'^2 - \frac{3003}{8192}\Theta'^6\Phi^2\Delta'^3 \\
& - \frac{6435}{65536}\Theta'^3\Phi\Delta' + \frac{2431}{262144}\Theta'^{10},
\end{aligned}
$$

$$
\begin{aligned}
A_{11}\Delta'^{\frac{21}{2}} = {} & \frac{3}{16}\Theta\Delta^2\Delta'^8 - \frac{15}{32}\Theta'\Phi\Delta^2\Delta'^7 - \frac{15}{32}\Theta'\Theta^2\Delta\Delta'^7 - \frac{15}{32}\Phi^2\Theta\Delta\Delta'^7 \\
& - \frac{5}{32}\Phi\Theta^3\Delta'^7 + \frac{35}{128}\Theta'^3\Delta^2\Delta'^6 + \frac{35}{128}\Theta'^2\Theta^3\Delta'^6 + \frac{105}{64}\Theta'^2\Phi\Theta\Delta\Delta'^6 \\
& + \frac{35}{64}\Theta'\Phi^3\Delta\Delta'^6 + \frac{105}{128}\Theta'\Phi^2\Theta^2\Delta'^6 + \frac{35}{256}\Phi^4\Theta\Delta'^6 - \frac{315}{1024}\Theta'^4\Theta\Delta\Delta'^5 \\
& - \frac{315}{256}\Theta'^3\Phi^2\Delta\Delta'^5 - \frac{315}{256}\Theta'^3\Phi\Theta^2\Delta'^5 - \frac{315}{256}\Theta'^2\Phi^3\Theta\Delta'^5 - \frac{63}{256}\Theta'\Phi^5\Delta'^5 \\
& + \frac{693}{1024}\Theta'^5\Phi\Delta\Delta'^4 + \frac{693}{2048}\Theta'^5\Theta^2\Delta'^4 + \frac{3465}{2048}\Theta'^4\Phi^2\Theta\Delta'^4 \\
& + \frac{1155}{2048}\Theta'^3\Phi^4\Delta'^4 - \frac{429}{4096}\Theta'^7\Delta\Delta'^3 - \frac{3003}{4096}\Theta'^6\Phi\Theta\Delta'^3 \\
& - \frac{3003}{4096}\Theta'^5\Phi^3\Delta'^3 - \frac{6435}{65536}\Theta'^8\Theta\Delta'^2 - \frac{6435}{16384}\Theta'^7\Phi^2\Delta'^2 \\
& + \frac{12155}{131072}\Theta'^9\Phi\Delta' - \frac{4199}{524288}\Theta'^{11}.
\end{aligned}
$$

The condition that a tetrahedron could have two pairs of opposite edges lying on a quadric V and its four faces touching another quadric U was $A_8 = 0$,

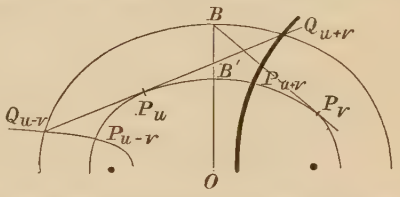
or, from the above, $8\Theta\Delta'^2 - 4\Theta'\Delta'\Phi + \Theta'^3 = 0$,

which agrees with the result given by Salmon (*Geometry of Three Dimensions*, § 207).

Note on the Derivation of Elliptic Function Formulæ from Confocal Conics. By John Griffiths, M.A.

[*Read Nov. 9th*, 1882.]

1. The ordinary formulæ for $\mathrm{sn}(u+v)$ and $\mathrm{cn}(u+v)$ may be derived from the following simple construction, viz., draw the tangent, as in the accompanying figure, at any point P_u on the inner of two confocal ellipses so as to intersect the outer curve in two points Q_{u-v}, Q_{u+v}, say; through each of the latter draw a confocal hyperbola cutting the inner ellipse in P_{u-v} and P_{u+v}; then the elliptic integral of the first kind, say $u-v$, becomes u as we pass from the point P_{u-v} to P_u, and similarly P_{u+v} corresponds to the function $u+v$, the period in each case being v, which is a constant quantity when we take two fixed ellipses.



2. The equation of the outer ellipse in terms of the foregoing elliptic parameter v may thus be found:—

Taking the eccentricity of the inner curve to be k and the semi-major axis as unity, the equation of this curve is $x^2+\frac{y^2}{k'^2}=1$, where $k^2+k'^2=1$; also, if we draw a tangent from B, an extremity of the minor axis of the outer ellipse, to the inner curve, and P_v be the point of contact, then the period $v=\int_0^{\mu}\frac{d\mu}{\sqrt{1-k^2\sin^2\mu}}$, μ being the eccentric angle of P_v, or $\mu=\mathrm{am}\,v$. Hence we may write for the equation of the outer ellipse $\frac{x^2}{\mathrm{dn}^2 v}+\frac{y^2}{k'^2}=\frac{1}{\mathrm{cn}^2 v}$.

The above construction enables us at once to form a quadratic whose roots $\mathrm{sn}(u+v)$, $\mathrm{sn}(u-v)$ are given in terms of $\mathrm{sn}\,u$ and $\mathrm{sn}\,v$.

For, the equation of the tangent at P_u is $x\,\mathrm{sn}\,u+\frac{y\,\mathrm{cn}\,u}{k'}=1$, and we have only to find the points where this line intersects the outer curve. Now, taking for the coordinates of a point on the latter ellipse

$$x=\frac{\mathrm{dn}\,v}{\mathrm{cn}\,v}\,\mathrm{sn}\,\sigma,\quad y=k'\frac{\mathrm{cn}\,\sigma}{\mathrm{cn}\,v},$$

we obtain

$$\frac{\mathrm{sn}\,u\,\mathrm{dn}\,v}{\mathrm{cn}\,v}\,\mathrm{sn}\,\sigma+\frac{\mathrm{cn}\,u\,\mathrm{cn}\,\sigma}{\mathrm{cn}\,v}=1,$$

where the roots of σ are $u-v$ and $u+v$. Hence, since $\mathrm{sn}^2\sigma+\mathrm{cn}^2\sigma=1$,

the equation in question gives, after some slight reductions,

$$(1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v)\,\mathrm{sn}^2\sigma - 2\,\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v\,.\,\mathrm{sn}\,\sigma + \mathrm{sn}^2 u - \mathrm{sn}^2 v = 0,$$

or
$$\mathrm{sn}\,(u-v)+\mathrm{sn}\,(u+v) = 2\,\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v \div (1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v),$$

$$\mathrm{sn}\,(u-v)\,\mathrm{sn}\,(u+v) = \mathrm{sn}^2 u - \mathrm{sn}^2 v \div (1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v)$$

(see Cayley's "Elliptic Functions," page 65). Similarly we have the expressions for $\mathrm{cn}\,(u-v)$ and $\mathrm{cn}\,(u+v)$.

By taking two points Q_u and Q_v on the outer ellipse, so related that the line joining them touches the inner curve at $P_{\frac{u+v}{2}}$, we derive, in like manner, from the above theory, the formulæ

$$\mathrm{cn}\,\frac{u-v}{2}\cos\frac{\mathrm{am}\,u+\mathrm{am}\,v}{2} = \mathrm{cn}\,\frac{u+v}{2}\cos\frac{\mathrm{am}\,u-\mathrm{am}\,v}{2},$$

$$\mathrm{cn}\,\frac{u-v}{2}\sin\frac{\mathrm{am}\,u+\mathrm{am}\,v}{2} = \mathrm{sn}\,\frac{u+v}{2}\cos\frac{\mathrm{am}\,u-\mathrm{am}\,v}{2}\,\mathrm{dn}\,\frac{u-v}{2}.$$

It may be also remarked that by taking different outer ellipses we have a geometrical construction for $\mathrm{sn}\,(u+v+w+\ldots)$, $\mathrm{cn}\,(u+v+w+\ldots)$, when $\mathrm{sn}\,u$, $\mathrm{sn}\,v$, ..., $\mathrm{cn}\,u$, $\mathrm{cn}\,v$, ..., &c. are given.

3. The above method of deriving the elliptic function formulæ suggests some rather interesting problems; for example, the division of the transcendents

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad \text{and} \quad E = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\;d\theta$$

into an integer number of equal parts depends upon the construction of an outer confocal ellipse, so that a polygon of a given number of sides circumscribing the inner curve has its vertices on the outer one. In fact, the ratio of the minor axes of the curves is $\mathrm{cn}\,\frac{2K}{n} : 1$, when the number of sides of the polygon in question is n. For example, if $n=4$, we find that a rectangle circumscribing the ellipse $x^2+\frac{y^2}{k'^2}=1$, and having its sides parallel to the principal axes, is also inscribed in the confocal ellipse $x^2+\frac{y^2}{k'}=1+k'$, and hence that

$$\mathrm{cn}\,\frac{K}{2} = \frac{k'}{\sqrt{k'(1+k)}}, \quad \text{or} \quad \mathrm{cn}\,\frac{K}{2} = \sqrt{\frac{k'}{1+k'}},$$

the value given in Cayley's "Elliptic Functions," page 74.

It further appears, from my recent extension of Graves' theorem, that when two given confocal ellipses are so related to each other that a polygon of a given number of sides can be inscribed in the one and circumscribed to the other, the perimeter must be of constant length.

If p is the perimeter of such polygon, of n sides, we have

$$p = 4E + 2n\,(\text{tangent } BP_v - \text{arc } B'P_v) \quad \text{(see figure).}$$

[In further explanation of the assumption that the elliptic integral of the first kind has a period v added to or subtracted from it as the passage is made from a point P_u to another point P_{u+v}, constructed as above, it may be noticed that Q_{u+v} and P_{u+v}, for instance, are corresponding points on the two ellipses, *i.e.*, their eccentric angles or amplitudes are equal. Let us assume, for a moment, that corresponding to P_{u+v} there is the integral $u+v=\int_0^{\psi}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$, then the coordinates of P_{u+v} and Q_{u+v} are

$$\operatorname{sn}(u+v),\quad k'\operatorname{cn}(u+v);\quad \frac{\operatorname{dn} v}{\operatorname{cn} v}\operatorname{sn}(u+v),\quad \frac{k'}{\operatorname{cn} v}\operatorname{cn}(u+v).$$

Again, to prove the existence of the period v; let θ, ϕ, ψ be the respective amplitudes corresponding to Q_{u-v}, P_u, Q_{u+v}; then taking, for the sake of symmetry, the semiaxes of the inner and outer curves to be a', b', and a, b, we have, since a tangent is drawn to the inner ellipse,

$$\frac{a}{a'}\sin\phi\sin\theta+\frac{b}{b'}\cos\varphi\cos\theta=1,$$

$$\frac{a}{a'}\sin\phi\sin\psi+\frac{b}{b'}\cos\phi\cos\psi=1,$$

and also

$$a^2-b^2=a'^2-b'^2;$$

whence, by differentiation, (Jacobi's method, see Cayley's "Elliptic Functions"), there result the equations

$$\frac{d\theta}{\sqrt{a'^2\cos^2\theta+b'^2\sin^2\theta}}=\frac{d\phi}{\sqrt{a'^2\cos^2\varphi+b'^2\sin^2\varphi}}=\frac{d\psi}{\sqrt{a'^2\cos^2\psi+b'^2\sin^2\psi}}.$$

Now, if the point P_u be taken to coincide with B', an extremity of the minor axis of the inner curve, we have $\varphi=0$ and $\theta=-\mu$ simultaneously, where $\cos\mu=\frac{b'}{b}$; also $\phi=0$ and $\psi=+\mu$; for the points denoted by Q_{u-v} and Q_{u+v} will then be situated symmetrically to the left and right of the minor axis. Hence, remembering that P_{u-v} and P_{u+v} have the same amplitude, respectively, as Q_{u-v} and Q_{u+v}, we obtain, by integration,

$$\int_0^{\theta}\frac{d\theta}{\sqrt{a'^2\cos^2\theta+b'^2\sin^2\theta}}$$

$$=\int_0^{\phi}\frac{d\phi}{\sqrt{a'^2\cos^2\varphi+b'^2\sin^2\varphi}}-\int_0^{\mu}\frac{d\mu}{\sqrt{a'^2\cos^2\mu+b'^2\sin^2\mu}},$$

$$\int_0^{\psi}\frac{d\psi}{\sqrt{a'^2\cos^2\psi+b'^2\sin^2\psi}}$$

$$=\int_0^{\phi}\frac{d\phi}{\sqrt{a'^2\cos^2\varphi+b'^2\sin^2\phi}}+\int_0^{\mu}\frac{d\mu}{\sqrt{a'^2\cos^2\mu+b'^2\sin^2\mu}};$$

or, in other words, there is a period v, as assumed above.]

December 14th, 1882.

Prof. HENRICI, F.R.S., President, in the Chair.

Messrs. T. Woodcock, B.A., late Scholar of Christ's College, Cambridge; Hugh Fraser, B.A., late Scholar of Trinity Hall, Cambridge; Major Allan Cunningham, R.E.; and Captain P. A. Macmahon, R.A., were elected members.

The Chairman announced that, in consequence of ill-health, Mr. C. W. Merrifield, F.R.S., had been obliged to resign the office of Treasurer, and that the Council had elected Mr. A. B. Kempe, F.R.S., to undertake the duties of the office. Dr. Hirst spoke of the great interest the late Treasurer had always taken in the affairs of the Society, and of the great services he had rendered to it in the past; and moved that, under the circumstances of his enforced resignation, a vote of sympathy with Mr. Merrifield in his illness be passed by the Society, and the same signified to him by letter from the Secretaries. This proposition was carried unanimously.

Mr. Tucker presented an Album for cabinet likenesses to the Society.

The following communications were made:—

"On the Vibrations of a Spherical Shell," Prof. Lamb.

"On the Absolutely Least Periods of the Elliptic Functions," Prof. H. J. S. Smith.

"On certain Relations between Volumes of Loci of Connected Points," Mr. E. B. Elliott.

"Geometrical Proof of Griffiths's Extension of Graves's Theorem," Mr. J. J. Walker.

"On Polygons Circumscribed about a Tricuspidal Quartic," Mr. R. A. Roberts.

"Note on an Exceptional Case in which the Fundamental Postulate of Prof. Sylvester's Theory of Tamisage fails," Mr. J. Hammond.

"On certain Quartic Curves, which have a Cusp at Infinity, whereat the Line at Infinity is a Tangent," Mr. H. M. Jeffery.

The following presents were received:—

"The Mathematical Magazine, a Journal of Elementary Mathematics," Vol. i., No. 4, Oct. 1882; Erie, Pa., 1882.

"Nautical Almanac for 1886," 8vo; London, 1882.

"Educational Times," for December.

"Bulletin de la Société Mathématique de France," Tome x., No. 6.

"Bulletin des Sciences Mathématiques et Astronomiques," 2nd Series, Tome vi.; July, 1882.

"Bulletin Astronomique et Météorologique de l'Observatoire Impérial de Rio de Janeiro," September, 1882, No. 9.

"Beiblätter zu den Annalen der Physik und Chemie," Band vi., Stücke 10 & 11.

"Crelle," Dreiundneunzigster Band, drittes Heft.

"Mathematische Annalen," &c., Seperatabdruck aus dem xxi. Bande, pp. 141—218.

"American Journal of Mathematics," Vol. v., No. 1; Baltimore, 1882.

On the Vibrations of a Spherical Shell. *By* Horace Lamb, M.A.

[*Read Dec.* 14*th*, 1882.]

1. The methods employed in a former communication, "On the Vibrations of an Elastic Sphere,"* can of course be readily adapted to the case of a shell bounded by two concentric spherical surfaces; the chief modification necessary being that we must now associate with every term of our formulæ involving a solid harmonic of degree n, a precisely similar term in which n is replaced by $-n-1$. The most interesting case is when the thickness of the shell is infinitely small compared with the radius. The results of the investigation then assume a very simple form, and may be stated as follows:—

The fundamental modes of vibration fall as before into two classes.

In the modes of the *First Class*, the motion at every point of the shell is wholly tangential. In the n^{th} species of this class, the lines of motion are the contour lines of a surface harmonic S_n, and the amplitude of the vibration at any point is proportional to the value of $dS_n/d\epsilon$, where $d\epsilon$ is the angle subtended at the centre by a linear element drawn on the surface of the shell at right angles to the contour line passing through that point. The frequency $(p/2\pi)$ is determined by the equation

$$k^2a^2 = (n-1)(n+2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where a is the radius of the shell, and $k^2 = p^2\rho/\mathfrak{n}$, if ρ denote the density, and $\mathfrak{n}$ the rigidity, of the substance.

In the vibrations of the *Second Class*, the motion is partly radial, and partly tangential. In the n^{th} species of this class the amplitude of the radial component is proportional to S_n, a surface harmonic of order n. The tangential component is everywhere at right angles to the contour lines of the harmonic S_n on the surface of the shell, and its amplitude is proportional to $\Lambda\, dS_n/d\epsilon$, where Λ is a certain constant, and $d\epsilon$ has the same meaning as before. It will appear further on (§ 3) that

$$\Lambda = -\frac{k^2a^2-4\gamma}{2n(n+1)\gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where k retains its former meaning, and $\gamma = (1+\sigma)/(1-\sigma)$, σ denoting, as before, Poisson's ratio. Corresponding to each value of n there are *two* values of k^2a^2, given by the equation

$$k^4a^4 - k^2a^2\{(n^2+n+4)\gamma + n^2+n-2\} + 4(n^2+n-2)\gamma = 0 \ldots (3).\dagger$$

---

* *Proceedings*, Vol. xiii., pp. 189—212.

† The case $n=0$ is exceptional. The vibrations are then purely radial, and the frequency is determined by $k^2a^2 = 4\gamma$.

Of the two roots of this equation, one is $<$ and the other $>4\gamma$. It appears, then, from (2) that the corresponding fundamental modes are of quite different characters. The mode corresponding to the lower root is always the more important.

When $n=1$, the values of k^2a^2 are 0 and 6γ. The zero root corresponds to a motion of translation of the shell as a whole parallel to the axis of the harmonic S_1. In the other mode the radial motion is proportional to $\cos\theta$, where θ is the colatitude measured from the pole of S_1; the tangential motion is along the meridian, and its amplitude (estimated in the direction of θ increasing) is proportional to $\frac{1}{2}\sin\theta$.

When $n=2$, the values of ka corresponding to various values of σ are given by the following table :—

$\sigma=0$	$\sigma=\frac{1}{4}$	$\sigma=\frac{3}{10}$	$\sigma=\frac{1}{3}$	$\sigma=\frac{1}{2}$
1·120	1·176	1·185	1·190	1·215
3·570	4·391	4·601	4·752	5·703

The most interesting variety is that of the zonal harmonic. Making $S_2=(3\cos^2\theta-1)/2$, we see that the polar diameter of the shell alternately elongates and contracts, whilst the equator simultaneously contracts and expands respectively. In the mode corresponding to the lower root, the tangential motion is towards the poles when the polar diameter is lengthening, and *vice versâ*. The reverse is the case in the other mode. We can hence understand the great difference in frequency.

2. To explain the manner in which the foregoing results were obtained, it may be sufficient to take the vibrations of the First Class. Omitting the time factor, the component displacements parallel to a system of rectangular axes having the centre of the shell as origin are given by the formulæ

$$\alpha=\psi_n(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\chi_n+\psi_{-n-1}(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\chi_{-n-1},\ \&c.,\ \&c. \quad \ldots\ldots\ldots(4),$$

where χ_n, χ_{-n-1} are solid harmonics of the algebraical degrees indicated by the suffixes, and $\psi_n(kr)$ is defined by [3].* The surface conditions [18] become

$$\left(r\frac{d}{dr}-1\right)\alpha=0,\ \&c.,\ \&c.,$$

whence

$$P_n\chi_n+P_{-n-1}\chi_{-n-1}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

* The square brackets are used to distinguish references to the former paper.

where $$P_n = kr\psi_n'(kr) + (n-1)\,\psi_n(kr).$$

The equation (5) is to hold for $r=a$ and for $r=b$, where a and b respectively denote the inner and outer radii of the shell, which is not yet assumed to be infinitely thin. Eliminating the surface harmonics involved in the two equations of the form (5), we are led to the following equation to determine k :—

$$\begin{vmatrix} P_n a^n, & P_{-n-1} a^{-n-1} \\ \&c. & \&c. \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

The second row is to be obtained from the first by writing b for a throughout. Passing now to the case of an infinitely thin shell, we write, in the second row, $b = a+da$, subtract the first row, and divide out by da. Now

$$a\frac{d}{da}(P_n a^n) = a^n\{k^2a^2\psi_n''(ka) + 2nka\psi_n'(ka) + n(n-1)\psi_n(ka)\}.$$

But, from [4] and [5], we have, identically,

$$k^2a^2\psi_n''(ka) + (2n+2)\,ka\psi_n'(ka) + k^2a^2\psi_n(ka) = 0.$$

Hence $$a\frac{d}{da}(P_n a^n) = a^n\{2ka\psi_n'(ka) + (k^2a^2 - n\,.\,n-1)\,\psi_n(ka),$$

and (6) becomes, after obvious reductions,

$$\begin{vmatrix} ka\psi_n'(ka) + (n-1)\,\psi_n(ka), & \&c. \\ (k^2a^2 - .n-1\,.\,n+2)\,\psi_n(ka), & \&c. \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots (7),$$

where the second column is to be obtained from the first by changing n into $-n-1$. Since this change leaves the product $(n-1)(n+2)$ unaltered, the left-hand side of (7) contains $k^2a^2-(n-1)(n+2)$ as a factor. It will be shown immediately that

$$ka\psi_n'(ka)\,\psi_{-n-1}(ka) - ka\psi_{-n-1}'(ka)\,\psi_n(ka) + (2n+1)\,\psi_n(ka)\,\psi_{-n-1}(ka) = 2n+1 \ldots\ldots\ldots (8),$$

so that the other factor is $2n+1$. Hence

$$k^2a^2 = (n-1)(n+2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The relation (8) follows from Helmholtz's theorem,* that if u, v be any two functions satisfying the equation $(\nabla^2+k^2)\,\phi = 0$, where $\nabla^2 = d^2/dx^2 + d^2/dy^2 + d^2/dz^2$, then

$$\iint\left(u\frac{dv}{dn} - v\frac{du}{dn}\right)dS = 0,$$

where the integration extends over the boundary of any region not

* See Rayleigh's *Sound*, t. 2, §§ 294, 327.

containing singular points of u or v, and dn denotes an element of the inwardly directed normal to dS. Making

$$u = \psi_n(kr) \cdot r^n T_n, \quad v = \psi_{-n-1}(kr) \cdot r^{-n-1} T_n,$$

where T_n is a surface harmonic of order n, and taking as the region in question the space included between two spheres described about the origin as centre, we readily find that the value of the left-hand side of (8) is independent of a. Making a infinitesimal, we see that this value is $2n+1$.

The frequencies of the various vibrations of the Second Class, for a shell whose inner and outer radii are a and b, are determined by the equation

$$\begin{vmatrix} A_n, & C_n, & B_{-n-1}, & D_{-n-1} \\ \&c., & \&c., & \&c., & \&c. \\ B_n, & D_n, & A_{-n-1}, & C_{-n-1} \\ \&c., & \&c., & \&c., & \&c. \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

where A_n, B_n, C_n, D_n are defined by [52], [53], [55], [56]. The second and fourth rows of the determinant are to be obtained from the first and third respectively by the substitution of b for a. The reduction of the determinant when $b = a + da$ is much facilitated by the use of (8), and of the similar relation in which h* is written for k, but the details of the work would occupy too much space to be reproduced here.

3. By way of verification, I have worked out the results of § 1 by a different method, confining myself, however, for simplicity, to the cases where the vibration is symmetrical about a diameter. Taking this diameter as axis of reference, let the equilibrium position of a point of the shell be defined in the usual way by the polar coordinates θ, ϕ; and let us suppose that in consequence of the displacement these become $\theta + \Theta$, $\phi + \Phi$, whilst the radius vector becomes $a + R$. If ds, ds_1 respectively denote the lengths of a linear element lying in the surface of the shell before and after the displacement, we find, since Θ, Φ, R are by hypothesis independent of ϕ,

$$ds_1^2 = ds^2 + 2\left(\frac{R}{a} + \frac{d\Theta}{d\theta}\right) a^2 d\theta^2 + 2 \sin\theta \frac{d\Phi}{d\theta} \cdot a d\theta \cdot a \sin\theta d\phi$$

$$+ 2\left(\frac{R}{a} + \Theta \cot\theta\right) a^2 \sin^2\theta \, d\phi^2.$$

Hence, if e, f denote the elongations along the meridian and the parallel of latitude, respectively, and c the shear parallel to the tan-

* h is defined by [21].

gent plane, we have

$$\left.\begin{aligned} e &= \frac{R}{a} + \frac{d\Theta}{d\theta} \\ f &= \frac{R}{a} + \Theta \cot\theta \\ c &= \sin\theta \frac{d\Phi}{d\theta} \end{aligned}\right\} \quad \text{.....(10).}$$

If g denote the elongation in the direction of the thickness, the condition that there is no stress perpendicular to the surface gives

$$(\mathfrak{m}+\mathfrak{n})\, g + (\mathfrak{m}-\mathfrak{n})\,(e+f) = 0 \quad \text{.....(11).*}$$

Substituting the value of g, hence obtained, in the general expression† for the potential energy (w) per unit volume, we obtain

$$w = \frac{2\mathfrak{n}}{\mathfrak{m}+\mathfrak{n}} \{\mathfrak{m}e^2 + \mathfrak{m}f^2 + (\mathfrak{m}-\mathfrak{n})\, ef\} + \tfrac{1}{2}\mathfrak{n}c^2,$$

or, writing γ for $(3\mathfrak{m}-\mathfrak{n})/(\mathfrak{m}+\mathfrak{n})$,

$$w = \mathfrak{n}\left\{\frac{\gamma+1}{2}(e^2+f^2) + (\gamma-1)\, ef + \tfrac{1}{2}c^2\right\} \quad \text{.....(12).}$$

To obtain the vibrations of the First Class, we put $R=0$ and (in consequence of the restriction as to symmetry) $\Theta = 0$. Hence $e = 0$, $f = 0$ and

$$w = \tfrac{1}{2}\mathfrak{n} \sin^2\theta \left(\frac{d\Phi}{d\theta}\right)^2.$$

The variational equation of motion, then, is

$$\rho\tau \int_0^\pi a \sin\theta \ddot{\Phi} \,.\, a\sin\theta\, \delta\Phi \,.\, 2\pi a^2 \sin\theta\, d\theta + \delta V = 0,$$

where ρ is the density, τ the thickness of the shell, and V is the potential energy of the deformation, viz.,

$$V = 2\pi a^2 \tau \int_0^\pi w \sin\theta\, d\theta \text{.....(13).}$$

The usual method then leads to the equation

$$\rho a^2 \sin^3\theta \ddot{\Phi} = \mathfrak{n} \frac{d}{d\theta}\left(\sin^3\theta \frac{d\Phi}{d\theta}\right).$$

Hence, if $\Phi \propto \sin pt$, we have, writing $p^2\rho/\mathfrak{n} = k^2$, $\cos\theta = \mu$,

$$\frac{d}{d\mu}\left\{(1-\mu^2)^2 \frac{d\Phi}{d\mu}\right\} + k^2 a^2 (1-\mu^2)\, \Phi = 0,$$

---

* $\mathfrak{m}$ is a constant such that $\mathfrak{m} - \frac{4}{3}\mathfrak{n}$ is the resilience of volume.

† Thomson and Tait, *Natural Philosophy*, § 695.

or $$(1-\mu^2)\frac{d^2\Phi}{d\mu^2}-4\mu\frac{d\Phi}{d\mu}+k^2a^2\Phi=0 \quad \text{.....(14).}$$

But the well-known equation satisfied by the zonal harmonic Q_n yields on differentiation

$$(1-\mu^2)\frac{d^3Q_n}{d\mu^3}-4\mu\frac{d^2Q_n}{d\mu^2}+(n-1)(n+2)\frac{dQ_n}{d\mu}=0 \quad \text{.....(15).}$$

Hence (14) is satisfied by

$$\Phi=\frac{dQ_n}{d\mu} \quad \text{.....(16).}$$

provided $$k^2a^2=(n-1)(n+2) \quad \text{.....(1).}$$

It may be shown that (16) and (1) together constitute the only solution of (14) which is finite all over the shell.

In the symmetrical vibrations of the Second Class, we have $\Phi=0$, $c=0$. Hence the variational equation of motion is

$$\rho\tau\int_0^\pi(\ddot{R}\delta R+a^2\ddot{\Theta}\delta\Theta)\,2\pi a^2\sin\theta\,d\theta+2\pi a^2\tau\int_0^\pi\delta w\sin\theta\,d\theta=0 \quad \text{...(17),}$$

where $\delta w=\mathfrak{n}\,[\{(\gamma+1)e+(\gamma-1)f\}\,\delta e+\{(\gamma-1)e+(\gamma+1)f\}\,\delta f]$.

Assuming that R and Θ both $\propto \sin pt$, introducing the values of e and f from (10), and writing as before $p^2\rho/\mathfrak{n}=k^2$, we find, by equating to zero the coefficient of δR in (17),

$$k^2a^2\frac{R}{a}\sin\theta=2\gamma\left(2\frac{R}{a}\sin\theta+\Theta\cos\theta+\sin\theta\frac{d\Theta}{d\theta}\right) \quad \text{.....(18).}$$

Again, equating (after a partial integration) the coefficient of $\delta\Theta$ to zero,

$$k^2a^2\Theta\sin\theta=-\frac{d}{d\theta}\left[\left\{2\gamma\frac{R}{a}+(\gamma+1)\frac{d\Theta}{d\theta}+(\gamma-1)\Theta\cot\theta\right\}\sin\theta\right]$$
$$+2\gamma\frac{R}{a}\cos\theta+(\gamma-1)\frac{d\Theta}{d\theta}\cos\theta+(\gamma+1)\Theta\frac{\cos^2\theta}{\sin\theta}.$$

This reduces to

$$k^2a^2\Theta\sin\theta=-\frac{2\gamma}{a}\frac{dR}{d\theta}\sin\theta-(\gamma+1)\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$
$$-2\Theta\sin\theta+(\gamma+1)\frac{\Theta}{\sin\theta} \quad \text{.....(19).}$$

Equations (18) and (19) may be written

$$(k^2a^2-4\gamma)\frac{R}{a}\sin\theta=2\gamma\frac{d}{d\theta}(\sin\theta\,.\,\Theta) \quad \text{.....(20),}$$

$$2\frac{\gamma}{a}\frac{dR}{d\theta}\sin\theta=-(k^2a^2+2)\sin\theta\,.\,\Theta$$
$$-(\gamma+1)\sin\theta\frac{d}{d\theta}\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\,.\,\Theta)\right\} \quad \text{.....(21).}$$

Now the zonal harmonic Q_n satisfies the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dQ_n}{d\theta}\right)+n(n+1)\,Q_n=0.$$

Hence (20) and (21) are satisfied by

$$\frac{R}{a}=Q_n,\quad \Theta=\Lambda\frac{dQ_n}{d\theta},$$

provided $$k^2a^2-4\gamma=-2\gamma\,.\,n(n+1)\,\Lambda \quad\ldots\ldots\ldots\ldots\ldots\ldots(22),$$

and $$2\gamma=\{(\gamma+1)\,n(n+1)-(k^2a^2+2)\}\,\Lambda \quad\ldots\ldots\ldots\ldots(23).$$

Eliminating Λ, we find

$$(k^2a^2-4\gamma)\{k^2a^2-n(n+1)\,\gamma-(n^2+n-2)\}=4n(n+1)\,\gamma^2,$$

which agrees with (3). When the values of ka have been found, the corresponding values of Λ are given by (22).

4. It is hardly necessary to point out that the vibrations of a *complete* spherical shell are in no way analogous to the flexural vibrations of an open shell or bowl, which have been discussed by Lord Rayleigh (*Proceedings*, Vol. xiii., p. 4). By a theorem due to Jellett, any deformation of a closed convex surface involves extension or contraction in some part of it. Hence, in our problem, that part of the potential energy which is due to the flexure may, if the shell be sufficiently thin, be neglected in comparison with that due to the extensions and contractions.

I find that a thin glass globe 20 centimètres in diameter should, in its gravest mode, make about 5350 vibrations per second.*

On Polygons circumscribed about a Tricuspidal Quartic.

By R. A. Roberts, M.A.

[*Read Dec. 14th*, 1882.]

1. In a recent number of the *Proceedings*, I arrived at some results concerning polygons circumscribed about a cuspidal cubic and inscribed in another curve. I propose, in this paper, to treat the tricuspidal quartic in a similar manner. As a tricuspidal quartic is the reciprocal of a nodal cubic, the problem is the same as to inscribe polygons in a nodal cubic; but I find it more convenient to consider the tricuspidal quartic.

* The data employed in the calculation are taken from Everett's *Units and Physical Constants*, § 61; viz., $n=2\,.\,40\times10^{11}$, $\rho=2\,.\,942$, $\sigma=\cdot258$.

2. If α, β, γ are the coordinates of a line, and x, y, z the coordinates of a point, the equation of a tricuspidal quartic may be reduced to the form $\alpha^3+\beta^3-\alpha\beta\gamma = 0$, where z is the double tangent; and then, for any tangent to the curve, we may put

$$\alpha = \vartheta, \quad \beta = \vartheta^2, \quad \gamma = 1+\vartheta^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

With this assumption, the equation of the intersection (xyz) of two tangents ϑ_1, ϑ_2 is found to be

$$(q^2-p)\,\alpha+(1-pq)\,\beta+q\gamma = 0,$$

where $p = \vartheta_1+\vartheta_2$, $q = \vartheta_1\vartheta_2$; hence

$$\rho x = q^2-p, \quad \rho y = 1-pq, \quad \rho z = q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If any relation exist between p and q, the point xyz will lie on a locus, and, to find where this locus meets the given curve, we substitute the values (2) in the equation of the curve in xyz coordinates:

$$27z^4-18xyz^2+4\,(x^3+y^3)\,z-x^2y^2 = 0,$$

whence we obtain $\quad (p^2-4q)\,(1+q^3+pq)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$

It appears then that, in general, the locus corresponding to any assumed relation between p and q will touch the given curve in as many points as the equation $1+q^3+pq = 0$ has roots.

3. Suppose we have $a\vartheta_1\vartheta_2+b\,(\vartheta_1+\vartheta_2)+c = 0$, or $aq+bp+c = 0$; then from (2) the locus of xyz is a conic, and from (3) we see that this conic has triple contact with the curve. Now the equations

$$\left.\begin{aligned} a_1\vartheta_2\vartheta_3+b_1\,(\vartheta_2+\vartheta_3)+c_1 &= 0 \\ a_2\vartheta_3\vartheta_1+b_2\,(\vartheta_3+\vartheta_1)+c_2 &= 0 \\ a_3\vartheta_1\vartheta_2+b_3\,(\vartheta_1+\vartheta_2)+c_3 &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

will coexist for an infinite number of values of ϑ_1, ϑ_2, ϑ_3, if

$$a_2c_3+c_2a_3-2b_2b_3 = 0, \quad a_3c_1+c_3a_1-2b_1b_3 = 0, \quad a_1c_2+c_1a_2-2b_1b_2 = 0 \ldots(5).$$

Thus we see that we can circumscribe about the curve an infinite number of triangles, such that each vertex will move along a conic having triple contact with the curve. The equations (5) express that the pairs of tangents to the curve, at the points where it is cut by every two of the conics, divide the double tangent harmonically.

The equations (4) with the conditions (5) can also exist for an infinite number of values of n variables ϑ_1, ϑ_2, ... ϑ_n, when n is greater than four; we can thus have an infinite number of polygons of more than four sides circumscribed about the curve, such that each vertex moves along a conic having triple contact with the curve.

4. Let ϑ_1, ϑ_2 be connected by the relation

$$l\,(\vartheta_1+\vartheta_2)+m\vartheta_1\vartheta_2\,(\vartheta_1+\vartheta_2)-(a+b\vartheta_1\vartheta_2+c\vartheta_1^2\vartheta_2^2) \equiv \phi\,(\vartheta_1, \vartheta_2) = 0 \ldots..(6),$$

which gives $$p = \frac{a+bq+cq^2}{l+mq};$$

then from (2) we see that the locus of the intersection of the tangents is a nodal cubic, which from (3) touches the curve four times. Thus, exactly by the same method as that which I used in my former paper, an infinite number of polygons can be circumscribed about a tricuspidal quartic and inscribed in a nodal cubic which touches the quartic four times, if a further relation, depending on the number of sides of the polygon, is satisfied.

To find the parameters q_1, q_2 of the node of this cubic, we have $\phi(t, \vartheta_1) = 0$, $\phi(t, \vartheta_2) = 0$, and $1+t\vartheta_1\vartheta_2 = 0$, the condition that the three tangents $t, \vartheta_1, \vartheta_2$ should pass through a point; also $q_1 = t\vartheta_1$, $q_2 = t\vartheta_2$. Considering, then, ϑ_1, ϑ_2 as the roots of the equation $\phi(t, \vartheta) = 0$, we get

$$\vartheta_1+\vartheta_2 = \frac{mt^2-bt+l}{t(ct-m)}, \quad \vartheta_1\vartheta_2 = \frac{a-lt}{t(ct-m)},$$

the latter of which equations combined with $1+t\vartheta_1\vartheta_2 = 0$ gives $t = \frac{a-m}{l-c}$; thus q_1, q_2 are the roots of the equation

$$(l-c)(lm-ac)q^2+\{m(m-a)^2+l(l-c)^2+b(m-a)(l-c)\}q + (m-a)(lm-ac) = 0 \ldots\ldots\ldots(7).$$

Hence, if $lm-ac = 0$, the parameters of the node are 0 and ∞; and then from (2) we see that the node of the locus lies on the double tangent of the given curve. But $lm-ac = 0$ is the condition for the coexistence of the equations

$$\phi(\vartheta_1, \vartheta_2) = 0, \ \phi(\vartheta_2, \vartheta_3) = 0, \ \phi(\vartheta_1, \vartheta_2) = 0.$$

Hence, we infer that, if a nodal cubic V touch a tricuspidal quartic U four times, and the node of the cubic lie on the double tangent of the quartic, then an infinite number of triangles can be circumscribed about U and inscribed in V.

If quadrangles can be inscribed in V and circumscribed about U, $am^2+cl^2-abc = 0$, the simplest interpretation of which seems to be that V meets the double tangent of U in *two* points, the tangents from which to U intersect on V. This relation is obtained by supposing two sides of the quadrangle to coincide with the double tangent. When the equation (7) has equal roots, the node of the cubic becomes a cusp, and we obtain results which are the reciprocals of some already obtained in my former paper.

5. Let ϑ_1, ϑ_2 be connected by the relation

$$(\vartheta_1+\vartheta_2)^2-(\vartheta_1+\vartheta_2)(l+m\vartheta_1\vartheta_2)+a+b\vartheta_1\vartheta_2+c\vartheta_1^2\vartheta_2^2 = \phi(\vartheta_1, \vartheta_2) = 0 \ldots(8),$$

which gives $$p^2-(l+mq)p+a+bq+cq^2 = 0;$$

then, if α, β are the roots of the equation

$$4(a+bq+cq^2)-(l+mq)^2=0,$$

we may assume $\quad q=\dfrac{\alpha+\beta\lambda^2}{1+\lambda^2},\quad p=\dfrac{l'+2m'\lambda+n'\lambda^2}{1+\lambda^2},$

and, substituting these values in (2), we see that the locus is a unicursal quartic. This quartic touches the given curve six times at the points corresponding to

$$(\alpha+\beta\lambda^2)^3+(1+\lambda^2)^3+(1+\lambda^2)(\alpha+\beta\lambda^2)(l'+2m'\lambda+n'\lambda^2)=0.$$

Thus, as before, an infinite number of polygons can be circumscribed about a tricuspidal quartic U and inscribed in a unicursal quartic V touching U six times, if a further relation depending on the number of sides of the polygon is satisfied between U and V. If

$$1+b+ac-lm=0,$$

we have $\quad \phi(\vartheta_1,\ \vartheta_2)=\phi(\vartheta_2,\vartheta_3)=\phi(\vartheta_3,\vartheta_1)=0$, simultaneously,

and then there is evidently a triple point of V. Hence an infinite number of triangles can be circumscribed about a tricuspidal quartic U, and inscribed in a quartic V with a triple point which touches U six times. The quartic V is not limited in any further manner, as such a curve is determined by ten conditions, and we have four independent constants at our disposal.

If a node of V lie on the given curve, it will, evidently, become a cusp, and take the place of one of the points of contact. If the three nodes of V lie on U, V will become a tricuspidal quartic. Thus an infinite number of polygons can be circumscribed about a tricuspidal quartic U and inscribed in a tricuspidal quartic V; the two curves U and V having triple contact with one another, and being such that the cusps of V lie on U. There must be a further relation depending on the number of sides of the polygon, which must be greater than three. By reciprocation, we see that an infinite number of polygons can be inscribed in one nodal cubic and circumscribed about another nodal cubic, if the two curves are connected by certain relations.

6. Suppose we have $\quad \phi(\vartheta_1)=\phi(\vartheta_2)=\phi(\vartheta_3)=\phi(\vartheta_4)\quad \ldots\ldots\ldots\ldots(9),$

where $\quad \phi(\vartheta)=\dfrac{1}{\vartheta^2}(a\vartheta^4+b\vartheta^3+c\vartheta^2+d\vartheta+e),$

then the four corresponding tangents will evidently form a quadrilateral whose six intersections of sides lie on the same locus. Dividing $\phi(\vartheta_1)-\phi(\vartheta_2)$ by $\vartheta_1-\vartheta_2$, we get

$$(\vartheta_1+\vartheta_2)(a\vartheta^2\vartheta_2^2-e)-d\vartheta_1\vartheta_2+b\vartheta^2\vartheta_2^2=0,$$

whence $\quad p=\dfrac{q(d-bq)}{aq^2-e};$

and substituting this value of p in (2), we see that the locus is a unicursal quartic determined by the equations

$$\left.\begin{aligned} x &= aq^4+(b-e)\,q^2-dq \\ y &= bq^3+(a-d)\,q^2-e \\ z &= aq^3-eq \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Since from these equations (10) we have

$$ex+(a-d)\,z+q\,\{(b-e)\,z-ay\}=0,$$

it follows that this quartic has a triple point whose coordinates are $a\,(d-a)$, ae, $e\,(b-e)$. Putting q equal to 0 and ∞ successively in (10), we see that this curve passes through xz and yz, the points of contact of the double tangent of the given curve; also, from (3), it touches the given curve five times. Hence we infer that, if a quartic V with a triple point touch a tricuspidal quartic U five times and pass through the points of contact of the double tangent of U, an infinite number of quadrilaterals can be circumscribed about U so as to have their six intersections of sides on V. The two curves U and V are not connected by any further relation, for a quartic with a triple point is determined by ten conditions, and we have three independent constants at our disposal.

7. Let $$\phi(\vartheta)=\frac{\vartheta^3+a\vartheta^2}{b\vartheta+c},$$

then the equations $$\phi(\vartheta_1)\,\phi(\vartheta_2)=1,$$

$$\phi(\vartheta_2)\,\phi(\vartheta_3)=1,\ \phi(\vartheta_3)\,\phi(\vartheta_4)=1,\ \phi(\vartheta_4)\,\phi(\vartheta_1)=1,$$

being only equivalent to three conditions, will give rise to a series of quadrilaterals, of which the intersections (12), (23), (34), (41) will lie on the same locus. From $\phi(\vartheta_1)\,\phi(\vartheta_2)=1$, we obtain

$$p=\frac{q^3+a^2q^2-b^2q-c}{bc-aq^2},$$

from which we see that the locus is the unicursal quartic determined by the equations

$$\left.\begin{aligned} x &= aq^4+q^3+(a^2-bc)\,q^2-b^2q-c^2 \\ y &= q^4+a^2q^3+(a-b^2)\,q^2-c^2q-bc \\ z &= aq^3-bcq \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(11).$$

This quartic touches the given curve five times, and is in general trinodal. Since $\phi(\vartheta_1)-\phi(\vartheta_3)=0$,

$$\phi(\vartheta_2)-\phi(\vartheta_4)=0,$$

or, putting $\vartheta_1+\vartheta_3=p$, $\vartheta_1\vartheta_3=q$,

$$c\,(p^2-q)+bpq+abq+acp=0,$$

it can be seen that the locus of the intersections of the opposite sides of the quadrilaterals (13), (24) is a quartic with a triple point.

8. From (2) we shall have

$$\rho^2(x^2-2yz) = q^4+p^2-2q, \quad \rho^2(y^2-2zx) = 1+q^2(p^2-2q), \quad \rho^2z^2 = q^2,$$

or, putting $\quad \vartheta_1^2+\vartheta_2^2 = p^2-2q = p', \ \vartheta_1^2\vartheta_2^2 = q^2 = q',$

$$\rho^2(x^2-2yz) = q'^2+p', \ \rho^2(y^2-2zx) = 1+p'q', \ \rho^2z^2 = q' \ldots\ldots (12).$$

Now, suppose we have the relations

$$l(\vartheta_1^2\vartheta_2^2+\vartheta_2^2\vartheta_3^2+\vartheta_3^2\vartheta_1^2)+m(\vartheta_1^2+\vartheta_2^2+\vartheta_3^2)+n = 0, \ \vartheta_1^2\vartheta_2^2\vartheta_3^2 = 1,$$

then, eliminating ϑ_3, we get

$$l(q'^2+p')+m(1+p'q')+nq' = 0.$$

Now $\vartheta_1\vartheta_2\vartheta_3 = -1$ is the condition that three tangents should pass through a point, and corresponding tangents (whose chord of contact touches the curve) are connected by the relation $\vartheta = -\vartheta'$. Hence we see that an infinite number of quadrangles can be inscribed in the conic

$$l(x^2-2yz)+m(y^2-2zx)+nz^2 = 0 \ldots\ldots (13),$$

so as to have the six lines joining their vertices tangents to the given curve. These conics (13) are the reciprocals of conics which touch the sides of quadrilaterals having their six intersections of sides on a cubic. The latter system of conics was, I believe, first investigated by Professor Cayley (Liouville's *Journal*, Tome x., p. 102).

If we have $\vartheta_1^2\vartheta_2^2\vartheta_3^2 = 1$, and a relation of the second degree between the quantities $\vartheta_1^2+\vartheta_2^2+\vartheta_3^2$ and $\vartheta_1^2\vartheta_2^2+\vartheta_2^2\vartheta_3^2+\vartheta_3^2\vartheta_1^2$, the locus of xyz is a quartic curve. Hence we see that an infinite number of quadrangles can be inscribed in the quartic

$$a(x^2-2yz)^2+b(y^2-2zx)^2+cz^4+2fz^2(y^2-2zx)+2gz^2(x^2-2yz) + 2h(x^2-2yz)(y^2-2zx) = 0 \ldots\ldots (14),$$

so as to have the six lines joining their vertices tangents to the given curve. This quartic can, evidently, be generated as the envelope of conics of the form (13), and then the quadrangles are the points of contact. The locus of the centres of the quadrangles is the Jacobian of $x^2-2yz = 0$, $y^2-2zx = 0$, $z^2 = 0$, viz., the conic $z^2-xy = 0$ (Salmon's *Curves*, Art. 253). Hence, also, twelve double tangents of the quartic (14) are touched by the given curve.

9. The method which I have adopted above may be extended to curves of higher orders. One of the results may be mentioned here. If we take a general unicursal curve U of the n^{th} degree, the coordinates α, β, γ of the chord joining two points ϑ_1, ϑ_2 on the curve, will be proportional to functions of $\vartheta_1+\vartheta_2$ and $\vartheta_1\vartheta_2$ of the $(n-1)^{th}$ degree. If, then, ϑ_1, ϑ_2 are connected by an equation of the form $\phi(\vartheta_1, \vartheta_2) = 0$ (8), the chord will touch a general unicursal curve V of class $2(n-1)$. As at (3) it can be shown that V touches U in $6n-2$ points, and since a unicursal curve of the $(2n-2)^{th}$ class is determined by $6n-7$ conditions, and $\phi(\vartheta_1, \vartheta_2)$ contains five constants, it follows

that, when it is possible to inscribe in U a polygon of a given number of sides which shall be circumscribed about V, only one further condition is satisfied. The double tangents of V are evidently determined by the equations $\phi(\vartheta_1, \vartheta_2) = 0$, $\phi(\vartheta_1, \vartheta_3) = 0$, combined with the condition that the three points $\vartheta_1, \vartheta_2, \vartheta_3$ should lie on a line, and also by the equations $\phi(\vartheta_1, \vartheta_2) = 0$, $\phi(\vartheta_3, \vartheta_4) = 0$, combined with the two conditions that the four points $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ should lie on a line. The first condition gives $3(n-2)$ double tangents, and the second the remainder. In the case of the triangle, these $3(n-2)$ double tangents are replaced by $n-2$ triple tangents, and then no further condition is satisfied.

If the parameters ϑ_1, ϑ_2 of a node of U satisfy the equation $\phi(\vartheta_1, \vartheta_2) = 0$, V is evidently divisible by the equation of that node. Thus, for every node of U whose parameters satisfy $\phi(\vartheta_1, \vartheta_2) = 0$, we must subtract one from the class of V, and, it can be shown, two from the points of contact of U and V. We can show, also, that every double tangent of V which becomes an inflexional tangent, touches U. The above result will be found to include several that have been already obtained.

On certain Relations between Volumes of Loci of Connected Points.

By Mr. E. B. Elliott.

[*Read Dec. 14th,* 1882.]

1. It is known (*Messenger of Mathematics*, Vol. vii., p. 151) that when a straight line, either extensible or not, occupies a closed infinite series of positions and corresponding states of extension in a plane, the area enclosed by the path of any point of it may be expressed linearly in terms of those enclosed by the paths of three chosen points of reference on it, or, more usefully, in terms of two such areas and a relative area easily interpreted generally, but of especial simplicity when the moving straight line is inextensible. We know also that, if a lamina, subject or not to homogeneous strain, move in a plane, in like manner the area enclosed by the path of any point in it can be expressed linearly in terms of those given by three chosen ones of its points and certain relative areas. This theorem, though practically arrived at by Mr. Leudesdorf (*Messenger*, Vol. vii., p. 125; Vol. viii., p. 11), has not, I believe, been definitely stated by him or by others, except for the case of the kinematics of a *rigid* lamina. Its general form is at once obtained by interpreting generally, instead of as he does specially, the analytical result obtained in the second of his two papers referred to, and may be stated,—"If a lamina subject to homogeneous strain

occupy a closed cycle of positions in a plane, a single definite state of strain corresponding to each position, then, if (A), (B), (C) be the areas enclosed by the paths of three points of reference A, B, C in it, (a), (b), (c) the areas that would be traced out by radii from fixed points always equal and parallel to CB, AC, BA respectively, and (P) the area enclosed by the path of P whose areal coordinates with regard to the triangle A, B, C are always x, y, z, this last is given by the equation $(P) = x(A) + y(B) + z(C) - yz(a) - zx(b) - xy(c)$" ...(1).

Considering this equation for a moment before passing to the main subject of our paper, we see, putting (P) constant, that the locus of points P of the lamina which surround areas equal to any given one, is one of a system of concentric similar and similarly situated conics, which in particular (Kempe's *Theorem*), in the case where the lamina is rigid, become the concentric circles, or straight lines if n be zero,

$$k = x(A) + y(B) + z(C) - n\pi(a^2yz + b^2zx + c^2xy).$$

Reference to special triangles in the lamina naturally simplifies the result (1), and materially helps the geometrical interpretation of conclusions. For instance, let (A_1), (B_1), (C_1) be the areas corresponding to the vertices of a triangle self-conjugate with regard to the real or imaginary conic which is the locus of points surrounding zero areas, then the area given by the point whose coordinates with regard to this triangle are x, y, z, is found to be

$$(P_1) = x^2(A_1) + y^2(B_1) + z^2(C_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the vanishing of the yz, zx, and xy terms giving the area relations

$$(a_1) = (B_1) + (C_1), \quad (b_1) = (C_1) + (A_1), \quad (c_1) = (A_1) + (B_1).$$

The form this takes for the case of rigid kinematics is noteworthy. In virtue of the known values $n\pi a_1^2$, $n\pi b_1^2$, $n\pi c_1^2$ for (a_1), (b_1), (c_1), the values of (A_1), &c. are at once $n\pi b_1 c_1 \cos A_1$, &c., and the relation becomes $(P_1) = 2n\pi\Delta_1(x^2 \cot A_1 + y^2 \cot B_1 + z^2 \cot C_1)$ .....(3),

where Δ_1 is the area of the triangle.

Another convenient special triangle of reference is one inscribed in one of the locus-conics. The form taken by the relation is too immediate to need statement.

2. Again, it is familiar (*Messenger*, Vol. vii., p. 153) that, if a straight line occupy a complete doubly infinite series of positions and corresponding states of extension in space, so that the various points of it traverse entire closed surfaces, the volume of any one of such surfaces can be expressed linearly in terms of those of four chosen ones, or of three chosen ones and a relative volume. The present paper is an endeavour to complete a series of space theorems by re-

placing the straight line here, first by a plane and then by a space, just as in two dimensions the area-theorem on the kinematics of a lamina completes the series introduced by that on the kinematics of a rod.

3. Let then, first, A, B, C be three points whose positions have a (1, 1, 1) correspondence, and cover respectively three closed surfaces of volumes (A), (B), (C); and let it be required to express as simply as possible the volume surrounded by the positions of the point P in the plane of ABC, whose areal coordinates with regard to the triangle ABC are always x, y, z. Considering the plane of ABC as a lamina, the alterations of size and shape in whose parts are due to constantly varying homogeneous strain, such a point P is of course always the same one of the lamina.

Referred to rectangular axes fixed in space, let (p_1, q_1, r_1), (p_2, q_2, r_2), (p_3, q_3, r_3) be the coordinates in corresponding positions of A, B, C; and suppose points filling the interiors of the volumes under consideration to be connected just as points covering their bounding surfaces are; then the volume element of the locus of P is

$$d(P)=(x\,dp_1+y\,dp_2+z\,dp_3)(x\,dq_1+y\,dq_2+z\,dq_3)(x\,dr_1+y\,dr_2+z\,dr_3)\,..(4).$$

Expanded, this is a sum of multiples of 27 volume elements mostly hard to interpret. These 27 belong, however, to only ten classes, corresponding to the ten multiples x^3, y^3, z^3, x^2y, x^2z, y^2z, xy^2, xz^2, yz^2, xyz. It occurs, then, that the sum may be expressed as one of ten independent elements only, and that these ten may be so chosen as to be interpreted without difficulty. It is possible to take the ten suitably in a large variety of ways. They may be the volume elements of the loci of ten points in the lamina; or some of them may be relative volume elements. The ten here chosen, that the form of result may have as much symmetry as possible, are the volume elements of (A), (B), (C), those of (A'), (B'), (C') the loci of the middle points of sides of the triangle ABC, that of (G) the locus of the centroid of the triangle, and those of (B, C), (C, A), (A, B) the relative loci of the vertices B, C, A with regard to C, A, B respectively. If the coefficients of these elements in the expansion of $d(P)$ be a, b, c, $8a'$, $8b'$, $8c'$, $27g$, a'', b'', c'' respectively, we have, for determining them, the equations which express the identity of the right-hand side of (4) with the following:—

$$\begin{aligned}
&a\,dp_1dq_1dr_1+b\,dp_2dq_2dr_2+c\,dp_3dq_3dr_3\\
&+a'd\,(p_2+p_3)\,d\,(q_2+q_3)\,d\,(r_2+r_3)+b'd\,(p_3+p_1)\,d\,(q_3+q_1)\,d\,(r_3+r_1)\\
&+c'd\,(p_1+p_2)\,d\,(q_1+q_2)\,d\,(r_1+r_2)\\
&+gd\,(p_1+p_2+p_3)\,d\,(q_1+q_2+q_3)\,d\,(r_1+r_2+r_3)\\
&+a''d\,(p_2-p_3)\,d\,(q_2-q_3)\,d\,(r_2-r_3)+b''d\,(p_3-p_1)\,d\,(q_3-q_1)\,d\,(r_3-r_1)\\
&+c''d\,(p_1-p_2)\,d\,(q_1-q_2)\,d\,(r_1-r_2),
\end{aligned}$$

that is to say, (1) three equations of which the type is

$$x^3 = a+b'+c'+g-b''+c'',$$

(2) three like $x^2y = c'+g-c'',$

(3) three like $xy^2 = c'+g+c'',$

(4) the one $xyz = g$:

and these suffice to determine uniquely the coefficients desired; viz.,

$$g = xyz,$$

$$2a'' = yz^2-y^2z,\quad 2b'' = zx^2-z^2x,\quad 2c'' = xy^2-x^2y,$$

$$2a' = yz^2+y^2z-2xyz,\quad 2b' = zx^2+z^2x-2xyz,\quad 2c' = xy^2+x^2y-2xyz,$$

$$a = x^3+xyz-xy^2-xz^2,\quad b = y^3+xyz-yz^2-yx^2,\quad c = z^3+xyz-zx^2-zy^2.$$

Inserting and integrating throughout, we have then, finally,

$$\begin{aligned}(P) = {} & x\,(x^2+yz-y^2-z^2)\,(A)+y\,(y^2+zx-z^2-x^2)\,(B)\\ & +z\,(z^2+xy-x^2-y^2)\,(C)\\ & +4yz\,(y+z-2x)\,(A')+4zx\,(z+x-2y)\,(B')+4xy\,(x+y-2z)\,(C')\\ & +27xyz\,(G)\\ & -\tfrac{1}{2}yz\,(y-z)\,(B,C)-\tfrac{1}{2}zx\,(z-x)\,(C,A)-\tfrac{1}{2}xy\,(x-y)\,(A,B)\quad\ldots\ldots(5).\end{aligned}$$

It is to be observed that $(B, C) = -(C, B)$, and so for the other relative volumes.

4. Putting the expression (5) for (P) constant, we obtain the equation in areal coordinates x, y, z of the locus in the moving plane of points of it which surround volumes equal to a given volume. This locus is, then, a curve of the third order. This is true whatever the value of (P) be—positive, zero, or negative. The loci are, therefore, real curves for all values of (P). The only case of exception to these statements is that in which the range of positions is such that

$$(A) = (B) = (C) = (A') = (B') = (C') = (G),$$

and $$(B, C) = 0 = (C, A) = (A, B),$$

i.e. the case of no rotation and no strain for which

$$(P) = (x+y+z)^2\,(A) = (A),$$

and is necessarily constant for all points of the lamina.

Moreover, the form of equation (5) shows that the locus cubics have all common asymptotes, real or imaginary, and indeed have all three-point contact at their common points at infinity.

Our present theorem as to volumes does not seem to admit, as does the corresponding theorem of area description, of a general simplification of statement applicable to all cases where the kinematics under consideration is that of an *indeformable* plane. In some in-

stances the relative volumes (B, C), (C, A), (A, B) will be zeroes, as in many also where there is deformation; in others they will be spherical volumes $\pm\frac{4}{3}\pi a^3$, $\pm\frac{4}{3}\pi b^3$, $\pm\frac{4}{3}\pi c^3$. Among other difficulties in the way of a general estimation of the relative volumes, that of interpretation of sign is not the least considerable.

5. It is useful to consider how the cubic equation of the (P)-volume locus can be simplified by reference to particular triangles in the lamina. Now, in a cubic curve it is possible to inscribe a triangle such that, as well as its vertices, the middle points of its sides lie upon the curve; for, to express that the three vertices and three mid-points lie on the cubic, just six conditions are needed in six independent coordinates of the vertices, which can therefore be determined so as to satisfy them. Let, then, such a triangle $A_1 B_1 C_1$ inscribed in the cubic corresponding to volume zero, be taken as that of reference. (5) will become, if x, y, z be the coordinates of P with regard to it,

$$(P) = 27xyz\,(G_1) - \tfrac{1}{2}yz\,(y-z)\,(B_1, C_1) - \tfrac{1}{2}zx\,(z-x)\,(C_1, A_1) - \tfrac{1}{2}xy\,(x-y)\,(A_1, B_1) \quad \ldots\ldots (6),$$

The value of this result is lessened by the probability that the triangle $A_1 B_1 C_1$ need not be real. To the following special reference no such objection can apply. It is known that some one of the four triangles with regard to which the equation of a given cubic reduces to the canonical form $\alpha x^3 + \beta y^3 + \gamma z^3 + \delta xyz = 0$, must necessarily be real, viz., the real "Wendepunktsdreieck" each of whose sides connects one real with two conjugate imaginary points of inflexion of the cubic. Let us now refer to $A_2 B_2 C_2$, this "Wendepunktsdreieck" for the cubic whose points surround zero volume. We obtain immediately, by expressing the identity of the above canonical form with the result of putting $(P) = 0$ in (5), and replacing (A), &c. by (A_2), &c., the reference volumes corresponding to the present triangle, that

$$\alpha = (A_2), \quad \beta = (B_2), \quad \gamma = (C_2),$$

and by a little manipulation that

$$(A_2') = \tfrac{1}{8}\{(B_2) + (C_2)\}, \quad (B_2') = \tfrac{1}{8}\{(C_2) + (A_2)\}, \quad (C_2') = \tfrac{1}{8}\{(A_2) + (B_2)\},$$

$$(B_2, C_2) = (B_2) - (C_2), \quad (C_2, A_2) = (C_2) - (A_2), \quad (A_2, B_2) = (A_2) - (B_2);$$

and thus get finally, for the volume of the locus in space of P whose coordinates with regard to this triangle are x, y, z,

$$(P) = x^3 (A_2) + y^3 (B_2) + z^3 (C_2) + \{27\,(G_2) - (A_2) - (B_2) - (C_2)\}\, xyz \ldots (7).$$

Once more, let the triangle formed by the common asymptotes of the locus cubics be taken as that of reference, so that the equation of any of them [and in particular of the one for which the value of (P) is zero] takes the form $(\alpha x + \beta y + \gamma z)\,(x+y+z)^2 + \delta xyz = 0$. A like

method to that last used, it being noticed that $x+y+z=1$, produces for this case the result

$$(P)=x(A_3)+y(B_3)+z(C_3)+9\{3(G_3)-(A_3)-(B_3)-(C_3)\}xyz..(8),$$

with the connecting relations between the reference volumes

$$(B_3, C_3)=0=(C_3, A_3)=(A_3, B_3),$$

$$(A_3')=\tfrac{1}{2}\{(B_3)+(C_3)\},\ (B_3')=\tfrac{1}{2}\{(C_3)+(A_3)\},\ (C_3')=\tfrac{1}{2}\{(A_3)+(B_3)\}.$$

This triangle formed by the asymptotes may of course be real, or may have only a single side real, according to circumstances. Even in the latter case, the result (8), though in itself unintelligible, may lead to sound geometrical conclusions, as in the like case of the following article.

Yet one more special triangle of reference may be mentioned, that which gives the zero-locus cubic the form of equation

$$(\alpha x^3+\beta y^2+\gamma z^2)(x+y+z)+\delta xyz=0.$$

Referred to it, the equivalent of (5) is

$$(P)=x^2(A_4)+y^2(B_4)+z^2(C_4)+3\{9(G_4)-(A_4)-(B_4)-(C_4)\}xyz...(9),$$

and the connecting relations between the reference volumes are

$$(B_4, C_4)=0=(C_4, A_4)=(A_4, B_4),$$

$$(A_4')=\tfrac{1}{4}\{(B_4)+(C_4)\},\ (B_4')=\tfrac{1}{4}\{(C_4)+(A_4)\},\ (C_4')=\tfrac{1}{4}\{(A_4)+(B_4)\}.$$

This triangle of reference has its sides clearly parallel to those of the last.

6. In just the same way as (5) and its special forms (6), (7), (8), (9) have been obtained, we may find like formulæ determining any relative volume (P, Q), *i.e.*, the volume which would be generated by radii vectores from a fixed point always equal to and in the direction of the relative radius vector QP, P and Q being two points of the moving plane. The general value for (P, Q) thus found is, x, y, z and x', y', z' being the constant areal coordinates of P and Q respectively, merely the result of substituting in the right-hand member of (5), $x-x', y-y', z-z'$ instead of x, y, z. We thus obtain the following theorem like in form to that on absolute volumes:—*The locus of points in the moving plane which surround any the same relative volume with regard to any point of it is a cubic curve.*

The expressions for (P, Q) in terms of the constants of the special triangles of reference $A_1B_1C_1$, $A_2B_2C_2$ are in like manner obtained from the right-hand side of (6) and (7) by insertion of $x-x', y-y', z-z'$ for x, y, z. The cases of (8) and (9) where the values of (P) contain terms of the first or second degree, as well as cubic terms in x, y, z, are however different; for, instead of having unity or a power

of $x+y+z$ multiplying such terms, the expressions for (P, Q) will have the like power of $x-x'+y-y'+z-z'$ or zero. Thus the corresponding result to (8), the triangle of reference being that formed by the asymptotes of the set of cubics giving different values of (P), is

$$(P, Q) = 9\{3(G_3)-(A_3)-(B_3)-(C_3)\}(x-x')(y-y')(z-z').$$

A glance at this assures us of the following :—*The cubics which are the loci of points in the moving plane that give equal relative volumes with regard to a fixed point in that plane have common asymptotes, viz., parallels through that point to the common asymptotes of the cubics in the plane which are the loci of points giving equal absolute volumes.* Moreover, the result tells us that, if either $x=x'$, $y=y'$, or $z=z'$, the relative volume (P, Q) vanishes. Hence,—*There are in the moving plane three directions, of which all or only one may be real, such that any radius QP parallel to either of them generates a vanishing relative volume ; viz., the directions of the above asymptotes.*

The theorem of area description in plane kinematics, which corresponds to that of the present article, is that the locus of points of the varying lamina which surround equal relative values k with regard to a fixed one of it, is the conic with that point as centre, to use the same notation as in (1),

$$(a)(y-y')(z-z')+(b)(z-z')(x-x')+(c)(x-x')(y-y') = k;$$

and, in particular, if the lamina be indeformable, is a circle.

7. Let us now proceed to the general case of one space of three dimensions, supposed to change its position with two degrees of freedom in a fixed space so as to occupy an entirely closed doubly infinite cycle of positions, and so that the loci of its points include closed volumes in that fixed space. We suppose that the form of any part of it is not necessarily unalterable, but that there is a single definite state of homogeneous strain corresponding to each position. Let a tetrahedron of reference be taken in the varying, and three rectangular axes of p, q, r in the fixed space.

Considering points filling the interiors of the various loci to be connected by possible positions of the varying space just as those upon the surfaces themselves are, the element of internal volume of the locus in the fixed space of a point P of the moving one whose coordinates (tetrahedral volume ratio) with regard to the chosen tetrahedron are, for all its states of homogeneous strain, x, y, z, w, is

$$d(P) = (x\,dp_1+y\,dp_2+z\,dp_3+w\,dp_4)(x\,dq_1+y\,dq_2+z\,dq_3+w\,dq_4)$$
$$(x\,dr_1+y\,dr_2+z\,dr_3+w\,dr_4) \quad \ldots\ldots\ldots\ldots\ldots\ldots(10),$$

where (p_1, q_1, r_1), (p_2, q_2, r_2), (p_3, q_3, r_3), (p_4, q_4, r_4) are the coordinates with regard to the fixed axes of the four vertices A, B, C, D.

Multiplied out, this expression for $d(P)$ contains 64 elements mostly uninterpretable, but belonging to only 20 classes, each with one of the 20 homogeneous products of three dimensions of x, y, z, w for common coefficient. It can then, doubtless, be expressed linearly in terms of 20 intelligible elements. Those chosen here for trial are the elements of (A), (B), (C), (D) the volumes of the loci of the vertices, of (AB), (AC), (AD), (BC), (BD), (CD) those of the loci of middle points of sides, of (G_a), (G_b), (G_c), (G_d) those of the loci of centroids of faces, and of (A, B), (A, C), (A, D), (B, C), (B, D), (C, D) the volumes of the relative loci of the first vertex mentioned in each case with regard to the second; and it is seen as follows, that coefficients a_1, a_2, a_3, a_4, $8b_{12}$, $8b_{13}$, $8b_{14}$, $8b_{23}$, $8b_{24}$, $8b_{34}$, $27c_1$, $27c_2$, $27c_3$, $27c_4$, d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34} of these elements may be uniquely determined to make the linear expression in terms of them practicable. We have, in fact, to make the right-hand members of (10) identical with

$$\sum_{i=1}^{i=4} a_i\, dp_i\, dq_i\, dr_i + \sum_{i=1,\, j=i+1}^{i=j-1,\, j=4} b_{ij}\,(dp_i+dp_j)\,(dq_i+dq_j)\,(dr_i+dr_j)$$
$$+ \sum_{i=1}^{i=4} c_i\,(\Sigma\, dp - dp_i)\,(\Sigma\, dq - dq_i)\,(\Sigma\, dr - dr_i)$$
$$+ \sum_{i=1,\, j=i+1}^{i=j-1,\, j=4} d_{ij}\,(dp_i-dp_j)\,(dq_i-dq_j)\,(dr_i-dr_j)\,;$$

that is to say, we have merely to solve the 20 linear equations in 20 unknowns,—

$$\begin{aligned}
a_1+b_{12}+b_{13}+b_{14}+c_2+c_3+c_4+d_{12}+d_{13}+d_{14} &= x^3,\\
a_2+b_{12}+b_{23}+b_{24}+c_1+c_3+c_4-d_{12}+d_{23}+d_{24} &= y^3,\\
a_3+b_{13}+b_{23}+b_{34}+c_1+c_2+c_4-d_{13}-d_{23}+d_{34} &= z^3,\\
a_4+b_{14}+b_{24}+b_{34}+c_1+c_2+c_3-d_{14}-d_{24}-d_{34} &= w^3,\\
b_{12}+c_3+c_4-d_{12} &= x^2y,
\end{aligned}$$

and five exactly similar expressions equated to $x^2z, x^2w, y^2z, y^2w, z^2w$ respectively, $\qquad b_{12}+c_3+c_4+d_{12} = xy^2,$

with five like equations with xz^2, xw^2, yz^2, yw^2, zw^2 for right-hand sides, and the four

$$c_1 = yzw, \quad c_2 = xzw, \quad c_3 = xyw, \quad c_4 = xyz.$$

Hence at once, the c's being already given by the last four equations, the type of the d's is $\qquad d_{12} = \tfrac{1}{2}xy\,(y-x),$

that of the b's is $\qquad b_{12} = \tfrac{1}{2}xy\,(x+y-2z-2w),$

and the a's are
$$\begin{aligned}
a_1 &= x\,(x^2-y^2-z^2-w^2+yz+yw+zw),\\
a_2 &= y\,(y^2-x^2-z^2-w^2+xz+xw+zw),\\
a_3 &= z\,(z^2-x^2-y^2-w^2+xy+xw+yw),\\
a_4 &= w\,(w^2-x^2-y^2-z^2+xy+xz+yz).
\end{aligned}$$

We have, then, by integrating over all positions,

$$(P) = a_1(A) + a_2(B) + a_3(C) + a_4(D)$$
$$+ 8\{b_{12}(AB) + b_{13}(AC) + b_{14}(AD) + b_{23}(BC) + b_{24}(BD) + b_{34}(CD)\}$$
$$+ 27\{c_1(G_a) + c_2(G_b) + c_3(G_c) + c_4(G_d)\}$$
$$+ d_{12}(A, B) + d_{13}(A, C) + d_{14}(A, D) + d_{23}(B, C) + d_{24}(B, D) + d_{34}(C, D)$$
$$\ldots\ldots\ldots\ldots(11),$$

with the above found cubic expressions in x, y, z substituted for the coefficients, a result which upon putting $w = 0$ is seen to include, as it should, the previously obtained one (5).

Putting the value of (P) constant, we obtain that the locus in the moving space of points of it whose loci in the fixed one contain volumes equal to a given one, is a cubic surface. For all values of (P), positive, zero, and negative, there is then such a real locus surface. Moreover, the form of equation shows that all the cubic surfaces thus obtained as loci have contact of the closest possible kind at an infinite distance.

8. The most convenient special tetrahedron to take for that of reference in the moving space, is that real one with regard to which the cubic $(P) = 0$ reduces to the form

$$\alpha x^3 + \beta y^3 + \gamma z^3 + \delta w^3 + \alpha' yzw + \beta' xzw + \gamma' xyw + \delta' xyz = 0.$$

The reference volumes being now those corresponding to this tetrahedron, the vanishing of the coefficients of x^2y, xy^2, &c. gives us twelve conditions in these, viz.,

six like $\quad 4(AB) - \frac{1}{2}(A, B) - (B) = 0,$

and six like $\quad 4(AB) + \frac{1}{2}(A, B) - (A) = 0;$

whence $\quad 8(AB) = (A) + (B)$, &c., and $(A, B) = (A) - (B)$, &c.,

and the value of (P), x, y, z, w being now coordinates referred to this tetrahedron, reduces to

$$(P) = (A)x^3 + (B)y^3 + (C)z^3 + (D)w^3$$
$$+ 27\{(G'_a)yzw + (G'_b)xzw + (G'_c)xyw + (G'_d)xyz\}\ldots\ldots(12),$$

where $\quad 27(G'_a) = 27(G_a) - (B) - (C) - (D)$, &c.

Two other special tetrahedra of reference may be mentioned, viz., (1) one of those whose vertices and the mid-points of whose edges lie upon the cubic surface $(P) = 0$ (determined by the satisfaction of ten conditions among twelve quantities), referred to which (2) is replaced by

$$(P) = 27\{(G_a)yzw + (G_b)xzw + (G_c)xyw + (G_d)xyz\}$$
$$- \tfrac{1}{2}\Sigma\{(A, B)xy(x - y)\}\ldots\ldots\ldots(13),$$

and (2) that referred to which the cubic surface $(P) = 0$ takes an equation of the form

$$(ax+\beta y+\gamma z+\delta w)\,(x+y+z+w)^2+\alpha' yzw+\beta' xzw+\gamma' xyw+\delta' xyz = 0,$$

which enables us to reduce (11) to

$$(P) = x\,(A)+y\,(B)+z\,(C)+w\,(D)$$
$$+27\,(G''_a)\,yzw+(G''_b)\,xzw+(G''_c)\,xyw+(G''_d)\,xyz\}\ldots\ldots(14),$$

in which $27\,(G''_a) = 27\,(G_a)-9\,\{(B)+(C)+(D)\}$, &c.

9. The relative volume (P, Q) of the locus in the fixed space of one point $P\ (x, y, z, w)$ of the moving one with regard to another $Q(x', y', z', w')$ of it, is derived from (P) by the substitution for x, y, z, w of $x-x', y-y', z-z', w-w'$ in the general expression for it (11). Thus the locus of points giving, with regard to a specified one, a constant relative volume, is a cubic surface. Of the expressions (12), (13), (14), and others simplified by special reference, the terms below the third degree in x, y, z, w will have corresponding to them no terms in the corresponding expressions for (P, Q) for a like reason to that adduced in §6. The simplified form corresponding to (14) for relative volumes will assure us immediately of the fact, that there are six directions, not necessarily all real, of lines QP through any point Q of the moving space which sweep out zero relative volumes.

On certain Quartic Curves, which have a Cusp at Infinity, whereat the Line at Infinity is a Tangent. By Henry M. Jeffery.

[*Read Dec.* 14*th*, 1882.]

1. In a recent memoir, published in these *Proceedings* (Vol. xiii., No. 185), quartics were classified, which were met and touched by the line at infinity in four coincident points,

$$\kappa\alpha^4 = u_2.$$

I now proceed to the next division of quartics (Salmon's "Higher Plane Curves," p. 213), which are met and touched by the same line in three coincident and met in one other real point (see § 31), and may be thus denoted $\kappa\alpha^3\beta = u_2$.

Inasmuch as the three points may coincide in a cusp, triple point, or stapete-point, the title of the memoir is imperfect.

The quartics are usually unicusped ovals, pirum-shaped, but occasionally cardioidal, and are either unipartite or bipartite.

2. The satellite-conic ($u_2=0$) has double contact of the second order, real or imaginary, with the quartic in each case.

Two parameters κ, λ are used for the purposes of discrimination; while λ varies, the satellite-conics are concentric and coaxal, *i.e.* homothetic, $2\kappa x^3y = ax^2+2bxy+cy^2+2ex+2dy+\lambda$.

It will be convenient to consider three divisions of these quartics, elliptic, hyperbolic, and parabolic, according as $b^2 < > = ac$.

3. Classification in this memoir also depends on two discriminating curves, designated hereinafter Curve (D_1) and Curve (D_2).

The equation in (κ, λ) to Curve (D_1) expresses the mutual relation between κ and λ, when the quartics have double points, and is found as the eliminant of the equations of certain pole curves.

Curve (D_2) is determined by eliminating between the quartic and the conditions that two stationary points unite to form a folium-point or point of undulation.

4. By triple asymptote of the quartics is here meant any line, which meets the curve in two coincident points of the cusp at infinity, in whose direction the directions of three possible asymptotes are merged.

5. To determine the tangential equivalent equation to a group of quartics with triple and single asymptotes, as above defined.

Let the line ($x\xi+y\eta=1$) touch the quartic

$$2\kappa x^3y = ax^2+2bxy+cy^2+2ex+2dy+\lambda.$$

The condition of contact is $S^3 = 27T^2$,

if $$S = \tfrac{1}{12}\phi^2+\kappa\eta\,(2\lambda\xi\eta^2+3d\xi\eta+e\eta^2+c\xi+b\eta),$$

where ϕ denotes the ends of the asymptotes $(a\eta^2-2b\xi\eta+c\xi^2)$, and

$$\begin{aligned} T = &-\tfrac{1}{216}\phi^3-\tfrac{1}{12}\kappa\eta\,(-4\lambda\xi\eta^2+3d\xi\eta+e\eta^2+c\xi+b\eta)\,\phi \\ &-\tfrac{1}{4}\kappa^2\,(\lambda\eta^4+2d\eta^3+c\eta^2)-\tfrac{1}{2}\kappa\xi\eta^3\,(d\xi-e\eta)^2 \\ &+\kappa\xi\eta^3\,[(ce-bd)\,\xi+(ad-be)\,\eta]-\tfrac{1}{2}\kappa\,(b^2-ac)\,\xi\eta^3. \end{aligned}$$

The equivalent class-curve is defined by a nonic equation

$$S^3-27T^2=0;$$

since η^3 disappears as a factor, as was anticipated, for the cusp ($\eta=0$). If there are no other or higher singularities, the quartics in this group belong to Dr. Salmon's Genus III.

6. To determine the first discriminating curve, or the mutual relation, which subsists between the parameters, when there are critical quartics in this group. Curve (D_1).

Let the previous equation to the group be resumed

$$-2\kappa x^3 y + ax^2 + 2bxy + cy^2 + 2ex + 2dy + \lambda = 0 \equiv \phi(x, y).$$

For critical values $\frac{d\phi}{dx} = 0, \quad \frac{d\phi}{dy} = 0,$

$$3\kappa x^2 y = ax + by + e \quad \ldots\ldots (1),$$

$$\kappa x^3 = bx + cy + d \quad \ldots\ldots (2);$$

whence $$0 = ax^2 + 2bxy + cy^2 + 3ex + 3dy + 2\lambda,$$

or $$0 = ax^2 - cy^2 + 2ex + \lambda \quad \ldots\ldots (3),$$

since, from (1) and (2),

$$x(ax + by + e) = 3y(bx + cy + d) \quad \ldots\ldots (4).$$

This last conic (4) is the locus of nodes of all quartics in the group. Curve (D_1) is the eliminant of (2), (3), and (4) and is of the tenth degree.

Cor. 1.—If $d = e = 0$, or the quartic asymptotes intersect in the centre of the satellite-conic, Curve (D_1) is resolved into two hyperbolæ. If $b^2 = ac$, it is represented by one hyperbola. See §§ 12—15.

Cor. 2.—If $c = 0$, or the quartics have a triple point at infinity, the Curve (D_1) is the quintic eliminant of the equations

$$\kappa x^3 = bx + d, \quad ax^2 + 2ex + \lambda = 0.$$

Cor. 3.—If $a = b = c = 0$, or the satellite is resolved into the lines

$$1 = 0, \quad 2ex + 2dy + \lambda = 0,$$

the Curve (D_1) is a quartic with a stapete-point and a folium-point

$$\kappa\lambda^3 + 8de^3 = 0.$$

7. On the locus of nodes, which may occur in quartics of this group. The discriminant of this conic

$$ax^2 - 2bxy - 3cy^2 + ex - 3dy = 0$$

is $$3ad^2 - 2bde - ce^2.$$

If this be zero, the locus of nodes is resolved into two right lines,

$$(1)\ ex - 3dy = 0, \quad (2)\ \frac{ax}{e} + \frac{cy}{d} + 1 = 0.$$

Curve (D_1) is therefore resolved into three portions: a quintic, a parallel to the (κ) axis, and a quartic. For (1),

$$\kappa x^3 = \left(b + \frac{ce}{3d}\right) x + d, \quad \lambda + x^2 \left(a - \frac{1}{9}\frac{ce^2}{d^2}\right) + 2ex = 0.$$

The eliminant is a bicusped quintic. To the cusp in a finite position

there corresponds a cusped quartic.

For (2), $$\kappa x^3 = bx+cy+d = \frac{x}{e}(be-ad),$$

$$a\lambda = e^2+\left(a-\frac{ce^2}{d^2}\right)ay^2 = e^2+\frac{2c}{d}(be-ad)\,y^2.$$

Either $$x = 0,\quad cy+d = 0,\quad c\lambda = d^2,$$

or a bicusped quartic is derived as the eliminant of

$$\kappa ex^2 = be-ad,\quad 2c(be-ad)\,y^2 = d(a\lambda-e^2).$$

If the discriminant be not zero, this conic is an important guide in determining the several branches of Curve (D_1). In particular, the two or four limiting points should be obtained, at which the tangents are parallel to the coordinate axes of x and y. Such points often correspond to cusps in the group of quartics, inasmuch as points on the two sides of these limiting points are crunodes and acnodes in the quartics, which eventually blend in cusps. See §§ 19, 23, Figs. 3, 5.

8. On the double points or cusps, which occur in the first discriminating curve.

At such points $$\frac{d\kappa}{dx} = 0,\quad \frac{d\lambda}{dx} = 0.$$

From (3) and (4) of § 6,

$$ax+e = cy\frac{dy}{dx},\quad 2bx+3cy+3d = cx\frac{dy}{dx}.$$

The elimination of $\frac{dy}{dx}$ gives (4), the locus of nodes.

The actual position of the four singular points is found by differentiating (4), and combining the resulting equation with (4).

9. On the critical quartics in this group.

The cusp at infinity may be merged in a triple point, at which two of the branches may be imaginary; such quartics are trinodal in the nascent state.

The quartics may be also binodal, if (κ, λ) is a point on Curve (D_1); thus, besides the given cusp, quartics may have a crunode, an acnode, attached to, or detached from, the rest of the curve, or a second cusp, formed by the union of a crunode and acnode.

The quartics may be also trinodal. Besides the given cusp, there may be two nodes, or a node and a second cusp, as happens when (κ, λ) is the intersection of two branches of Curve (D_1), its position on each branch taking separate effect.

Two crunodes may unite in a tacnode, and two acnodes in a higher singularity.

10. To determine the second discriminating curve, or the mutual relation which subsists between the parameters, when quartics of this group possess points of undulation. Curve (D_2).

At a folium-point* or point of undulation, two stationary points are consecutive, so that both $\frac{d^2y}{dx^2} = 0$, and $\frac{d^3y}{dx^3} = 0$.

If these tests be applied to quartics of this group,

$$2\kappa x^3 y = ax^2 + 2bxy + cy^2 + 2ex + 2dy + \lambda,$$

$$\kappa\left(3x^2y + x^3\frac{dy}{dx}\right) = ax + by + e + (bx + cy + d)\frac{dy}{dx},$$

$$\kappa\left(6xy + 6x^2\frac{dy}{dx}\right) = a + 2b\frac{dy}{dx} + c\left(\frac{dy}{dx}\right)^2,$$

$$y + 3x\frac{dy}{dx} = 0.$$

Curve (D_2) is defined by eliminating x, y from the given quartic, and

$$8\kappa x^3 y = 3ax^2 + 2bxy - cy^2 + 3ex - dy,$$

$$36\kappa x^3 y = 9ax^2 - 6bxy + cy^2.$$

It is of the twelfth degree in (κ, λ).

The locus of folium-points in the group of quartics is the conic

$$9ax^2 + 30bxy - 11cy^2 + 27ex - 9dy = 0.$$

If its discriminant is zero, or

$$ad^2 + 10bde - 11ce^2 = 0,$$

the conic is resolved into two straight lines

$$3ex - dy = 0, \quad \frac{3ax}{e} + \frac{11cy}{d} + 9 = 0.$$

Curve (D_2) is also resolved into two portions, a quartic and a sextic.

11. On the real inflexions which occur in quartics of this group.

It is established by Zeuthen (*Math. Annalen*, Band vii., p. 411), that a quartic can be at most quadrifolium, and therefore have at most eight inflexions.

Since the ordinary quartics of this group are unicusped ovals, and since two inflexions are merged in a cusp, there may be six inflexions in quartics of this group. According as the point (κ, λ) passes through the several branches of Curve (D_2), two inflexions are lost. There may be six, four, or two inflexions. See Figs. 2, 4.

But if the cusp at infinity becomes a triple point, only four inflexions are possible, and there may be two or none. See § 28, Fig. 10.

* By "folium," Zeuthen understands the depression in the contour of a quartic oval, which is characterised by two inflexions and a bitangent. At a folium-point, or point of undulation, this depression is nascent.

To apply these processes, we will commence with simple cases.

12. To determine all the quartics in this group, when the asymptotes of the quartic meet in the centre of the satellite.

All these quartics are central, and are thus defined:—

$$2\kappa x^3 y = ax^2 + 2bxy + cy^2 + \lambda.$$

By § 6, the locus of nodes consists of two lines through the centre.

$$ax^2 - 2bxy - 3cy^2 = 0.$$

Curve (D_1) is defined by the elimination of x from the equations

$$(3\kappa x^2 - b)(\kappa x^2 - b) = ac,$$

$$\kappa^3 x^6 + (ac - b^2)\kappa x^2 + 2\kappa\lambda c = 0.$$

The locus of (κ, λ) consists of two hyperbolæ and the (κ) axis.

There are three divisions as $b^2 = < > ac$.

Parabolic Division.—If $b^2 = ac$,

$$(1)\ x = 0, \text{ and } \lambda = 0,$$

$$(2)\ 3\kappa x^2 = 4b,$$

and the sign of (κ) depends on that of (b).

Curve (D_1) is represented by one branch of the hyperbola

$$27\kappa\lambda + 32ab = 0.$$

13. Curve (D_2) also consists of two hyperbolæ.

By § 9, the folium-points of this sub-group (§ 12) lie in the two lines $9ax^2 + 30bxy - 11cy^2 = 0.$

The eliminant of this equation, the given quartic, and the derived cubic $8\kappa x^3 y = 3ax^2 + 2bxy - cy^2$

determines two hyperbolæ for the locus of (κ, λ).

Parabolic Division.—If $b^2 = ac$, the locus of folium-points is resolved into two lines

$$(1)\ 3ax = by, \quad (2)\ 3ax + 11by = 0.$$

The Curve (D_2) is thus resolved into the axis $(\lambda = 0)$, and a branch of the hyperbola $(11)^3\kappa\lambda + 96ab = 0,$

which is the eliminant of the two equations

$$\kappa x^3 = 4cy, \quad 9\lambda = 8cy^2 \text{ (which limits the sign of } \lambda).$$

14. Discussion of the quartics of this sub-group (§ 12) in the parabolic division.

(1) If $\lambda = 0$, there is a tacnode at the centre, if

$$2\kappa x^3 y = (ax - by)^2,$$

and a double acnode, if $2\kappa x^3 y = (ax + by)^2.$

If κ be negative, the converse is true. The class of all is the fifth; they retain two inflexions, four having been merged at the centre, and two at the cusp.

(2) $$2\kappa x^3 y = ax^2 + 2bxy + \frac{1}{a} b^2 y^2 + \lambda.$$

If a, b, c be all positive, crunodes can correspond to points (κ, λ) in the second quadrant only, since for critical values $3\kappa x^2 = 4b$, and $27\kappa\lambda + 32ab = 0$. Curve (D_1).

But, since the quartics are central, they are trinodal for critical values of κ, λ. For values of (κ, λ) above Curve (D_1), the curve is unipartite; for values below it, bipartite.

Points of undulation occur for values of (κ, λ) in the fourth quadrant. For values of (κ, λ) below Curve (D_2), the quartics have two inflexions; for all others, six.

15. Discussion of the quartics of this sub-group in the elliptic and hyperbolic divisions.

(1) If $\lambda = 0$, for this sub-group,

$$2\kappa x^3 y = ax^2 + 2bxy + cy^2.$$

The origin is a biflecnode or an acnode; and the class is the seventh. Two inflexions are merged at the centre, four inflexions remain.

Folium-points exist, if the following relations are compatible:

$$\begin{aligned} -\ 2\kappa x^3 y + \ ax^2 + 2bxy + cy^2 &= 0, \\ -\ 8\kappa x^3 y + 3ax^2 + 2bxy - cy^2 &= 0, \\ -36\kappa x^3 y + 9ax^2 - 6bxy + cy^2 &= 0. \end{aligned}$$

For this condition $$49ac = 81b^2.$$

(2) $$2\kappa x^3 y = ax^2 + 2bxy + cy^2 + \lambda.$$

Critical values depend on the reality of the lines of nodes (§ 11),

$$ax^2 - 2bxy - 3cy^2 = 0.$$

If $b^2 + 3ac > 0$, there are further singularities: hence, in the elliptic division, crunodes must be possible; in the hyperbolic division, they may or may not exist.

For critical values, the quartics are trinodal; they have a pair of crunodes, since they are central, besides the cusp at infinity.

For values of (κ, λ) above or below the Curve (D_2), the quartics have six or two inflexions; these last disappear, at the folium-point, when (κ, λ) is on Curve (D_2), the discriminating hyperbola.

If there are no critical values, when $b^2 + 3ac < 0$, the quartics have four inflexions. Folium-points, however, are possible, if

$$25b^2 + 33ac < 0.$$

16. To determine the two discriminating curves, when the satellite-conic may be resolved into two parallel lines. (Parabolic Division.)

Let this sub-group be thus denoted (Figs. 1, 2):—

$$2\kappa x^3 y = (ax+by+c)^2+\lambda.$$

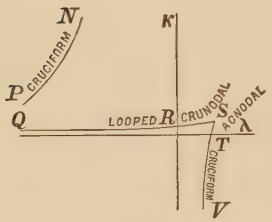


Fig. 1.

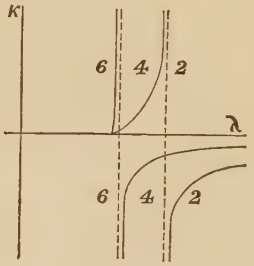


Fig. 2.

Curve (D_1) is, by § 6, the quintic eliminant of the quadratic and cubic

$$8b^2y^2+6bcy+\lambda+c^2=0, \quad 27\kappa b^2y^3-4a^3by-a^3c=0.$$

The locus of nodes is resolved into two lines,

$$ax-3by=0, \quad ax+by+c=0.$$

The first line determines the preceding quintic, the second the axis $(\lambda=0)$.

In the quintic a cusp is found at the point $8ax=24by=-9c$, determined by the conditions

$$\frac{d\kappa}{dy}=0, \quad \frac{d\lambda}{dy}=0.$$

To Curve (D_2), the equation is found by elimination. Since the locus of folium-points is defined by

$$3ax+11by+9c=0,$$

after rejecting the factor $(3ax-by)$,

$$2\kappa x^3y=(ax+by+c)^2+\lambda=\tfrac{1}{18}(3ax-by)^2.$$

A cusp is at the intersection of $(ax+by+c=0)$ and $(3ax=by)$. In this point both curves (D_1) and (D_2) meet, when $\kappa=0$, $\lambda=0$, and the quartic is resolved into two pairs of coincident lines.

17. To obtain all the quartics which occur, when the satellite is resolved into two parallel straight lines.

(1) If $\lambda=0$, or the satellite conic consists of two coincident lines, the quartics of § 15 are bicusped, and bifolium.

If $\lambda=0$, and $27\kappa c^2=8a^2b$, which is a point on the quintic Curve (D_1), the quartic is bicusped and crunodal, with two inflexions.

(2) If the case, when λ is zero, is excluded, the Curve (D_1) (Fig. 1) will be a guide to the singular and non-singular quartics of the group.

If (κ, λ) be a point in the first quadrant, and on the portion RS, the corresponding quartic is crunodal, with a loop.

This curve separates into an ampullate or bipartite form (the oval replacing the loop), as (κ, λ) is above or below RS.

If (κ, λ) be a point in TS, the oval shrinks to a detached acnode. At the cusp S of the quintic (D_1), the corresponding quartic is bicusped, since the crunode is united with the acnode to form a cusp.

In the second quadrant, if (κ, λ) is on QR, the continuation of RS, the quartics are still crunodal with a loop, and the companion-curves are ampullate, until (κ, λ) reaches PN, when the quartics become cruciform, and if (κ, λ) is above PN, the branches separate. If (κ, λ) is below QR, the quartics are bipartite.

In the third quadrant, (κ, λ) has no critical value.

In the fourth quadrant, if (κ, λ) is on TV, the quartics are crunodal and cruciform; for values of (κ, λ) within or without TV, the branches separate in different ways.

Curve (D_2) is confined to the first and fourth quadrants, and meets Curve (D_1) only in the cusp (0, 0).

There is, therefore, great simplicity in the classification of the inflexions. All quartics of this sub-group have six, four, or two inflexions, according as (κ, λ) lies to the left, between, or to the right of the branches of Curve (D_2), as is shown in Fig. 2.

18. In the parabolic division, Curve (D_1) is resolved into two portions, if the discriminant of the locus of nodes (§ 7) be zero.

$$3ad^2 - 2bde - ce^2 = 0.$$

Since $b^2 = ac$, (1) $ad - be = 0$, (2) $3ad + be = 0$.

The case (1) has been discussed in §§ 16, 17.

In case (2), the quartics in this sub-group are thus denoted:

$$2\kappa x^3 y = ax^2 + 2bxy + \frac{b^2}{a} y^2 - \frac{bad}{b} x + 2dy + \lambda.$$

The locus of nodes (§ 7) is resolved into two lines,

$$ax + by = 0, \quad x = \frac{3b}{a} y + \frac{3d}{b}.$$

From the first line there is derived one quartic portion of Curve (D_1), with a stapete-point and a folium-point,

$$b^3 \kappa \lambda^3 = 216 a^3 d^4, \text{ since } \kappa x^3 = d, \ b\lambda = 6adx.$$

From the second line, the other portion is a bicuspidal quartic

$$2a \sqrt{\left(\frac{b}{3\kappa}\right)} + e = \frac{3}{2\sqrt{2}} \sqrt{(e^2 - a\lambda)},$$

since, by § 6,

$$a\lambda - e^2 + 8b^2y^2 = 0, \quad \kappa x^3 = bx + \frac{b^2}{a}y + d = \frac{4}{3}bx.$$

A third portion of Curve (D_1) is found, when $x = 0$, $b^2\lambda = ad^2$.

The figures are readily seen to be limiting forms of Fig. 3. Curve (D_2) is resolved into two portions, when $ad = be$, but not when

$$3ad + be = 0.$$

19. In the parabolic division, to determine the first discriminant curve in the most general case.

For the quartics in this sub-group,

$$2\kappa x^3 y = ax^2 + 2bxy + \frac{b^2}{a}y^2 + 2ex + 2dy + \lambda.$$

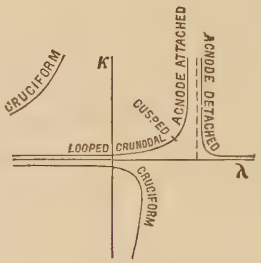


Fig. 3.

As in § 6, $\kappa x^3 = bx + \frac{b^2}{a}y + d$,

$$0 = ax^2 - \frac{b^2}{a}y^2 + 2ex + \lambda,$$

$$x(ax + by + e) = 3y\left(bx + \frac{b^2}{a}y + d\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

The elimination of x, y determines the locus of (κ, λ), or Curve (D_1). Curve (D_1) has three asymptotes, when

$$x = 0, \quad y = 0, \quad \kappa = \infty, \quad \lambda = 0,$$
$$x = 0, \quad by + e = 0, \quad \kappa = \infty, \quad a\lambda = e^2,$$
$$x = \infty, \quad y = \infty, \quad \kappa = 0, \quad \lambda = \infty.$$

Two limiting values of (x, y) (§ 7), in the locus of nodes (A), should be noticed, since for values of (κ, λ) on either side of one of the two corresponding points (κ, λ) of Curve (D_1) there may be a change from acnode to crunode through a cusp. There is no change which corresponds to the other limiting value of (x, y), in the cruciform portion of the (κ, λ) value.

20. To determine the various singular values of the quartics in the parabolic division, according as the parameters κ, λ are mutually related, as points on Curve (D_1). (Fig. 3.)

In the first quadrant, if (κ, λ) be on a branch of Curve (D_1) near the origin, the corresponding quartic is crunodal with a loop; but if (κ, λ) be distant, the quartics are acnodal, with an acnode attached to, or detached from, the quartics, as it lies on the further or nearer branch. At an intermediate position, where (x, y) is a limiting position of the locus of nodes, as above defined, the acnode and crunode

unite in a cusp. In the second quadrant, if (κ, λ) be on the lower branch of Curve (D_1), which is a continuation of the former branch, the quartics are crunodal and looped. If (κ, λ) be on the upper branch, the quartics are cruciform.

To points (κ, λ) situated in the third and fourth quadrants, the critical forms of the quadric are cruciform, without any alteration for the limiting value of (x, y) in the locus of nodes.

The non-singular companion-curves have unipartite forms near the cruciform quartics, and bipartite when (κ, λ) lies between parts of a branch, which yield looped crunodal and detached acnodal quartics.

21. To determine the various folium-points, in quartics of the parabolic division, as the parameters κ, λ are mutually related as points on Curve (D_2).

Let the equation be resumed for such quartics,

$$2\kappa x^3 y = ax^2 + 2bxy + \frac{b^2}{a} y^2 + 2ex + 2dy + \lambda$$

by § 10 $= \dfrac{1}{18a}(3ax - by)^2$.

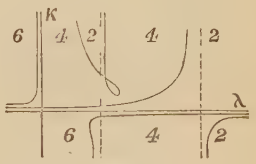


Fig. 4.

For the locus of folium-points,

$$9ax^2 + 30bxy - \frac{11}{a} b^2 y^2 + 27ex - 9dy = 0.$$

The elimination of x, y determines the locus of (κ, λ), or Curve (D_2). Curve (D_2) has four asymptotes, when

$$x = 0, \quad y = 0, \quad \kappa = \infty, \quad \lambda = 0.$$

$$x = 0, \quad y = -\frac{9ad}{11b^2}, \quad \kappa = \infty, \quad 242b^2\lambda = 243ad^2.$$

$$ax = -3e, \quad y = 0, \quad \kappa = \infty, \quad 2a\lambda = 3e^2.$$

$$x = \infty, \quad y = \infty, \quad \kappa = 0, \quad \lambda = \infty.$$

The Curves (D_1) and (D_2) do not intersect.

22. To determine the number of real inflexions in the quartics of the parabolic division, as the parameters (κ, λ) are mutually related with reference to the branches of Curve (D_2). (Fig. 4.)

The changes from six to four and two inflexions are best seen on the diagram (Fig. 4), where the figures 6, 4, 2 denote the number of inflexions which depend on the mutual values of (κ, λ). No change has been observed to take place as (κ, λ) passes through any branch of Curve (D_1), which passes between adjacent branches of Curve (D_2), as has been noticed in other divisions of Quartics. A change is noticed on either side of the (λ) axis.

23. To determine the singular values of quartics in the elliptic division, by the aid of the first discriminating curve—Curve (D_1).

The quartics of this sub-group are thus denoted: $2\kappa x^3 y = ax^2 + 2bxy + cy^2 + 2ex + 2dy + \lambda$, when $b^2 < ac$. The equations to Curve (D_1) are found in § 6. Four branches of Curve (D_1) meet in a double cusp, and two others in a single cusp. Three asymptotes occur, as in § 18. For related values of κ, λ, which correspond to points in Curve (D_1), the corresponding quartics are crunodal, acnodal, and (for two limiting values of x, y in the locus of nodes, § 7) cuspidal.

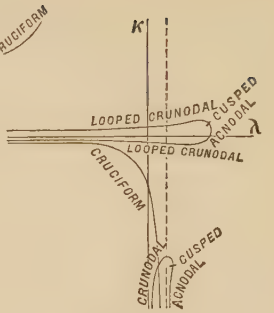


Fig. 5.

Corresponding to points (κ, λ) in the outer branches, the crunodal quartics are cruciform; for points in the two inner loops, the quartics are crunodal and looped or acnodal. The crunodes and acnodes unite in cusps at the two points above defined. In the lower loop of Curve (D_1) the acnodes are attached to their quartics, in the upper loop they are partly attached and partly detached. For points (κ, λ) within the upper loop the quartics are bipartite, but not for points (κ, λ) within the lower loop.

For the node (κ, λ) in Curve (D_1), in which two of its branches intersect, the corresponding quartic is trinodal, since the crunodal effect of each branch of Curve (D_1) is distinct.

The companion curves to the singular quartics are discussed in § 17.

24. To determine the number of points of inflexion of quartics in the elliptic division, by the aid of the second discriminating curve—Curve (D_2).

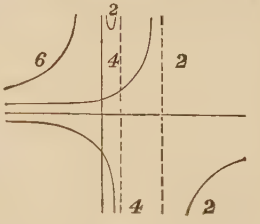


Fig. 6.

By reference to the diagram, the quartics are seen to have 6, 4, or 2 inflexions, as (κ, λ) lies with respect to the several branches of Curve (D_2). As (κ, λ) passes through a branch of Curve (D_2), two inflexions are merged in a folium-point. If (κ, λ) fall on any of the branches of Curve (D_1), which lie near or between branches of Curve (D_2), two inflexions are merged in a node; but, on either side of such a position, Curve (D_2) is the guide for any diversity.

25. To determine the singular values of quartics in the hyperbolic division, by the aid of the first discriminating curve—Curve (D_1).

In this division, if the quartic equation be assumed as usual,

$$2\kappa x^3 y = ax^2 + 2bxy + cy^2 + 2ex + 2dy + \lambda,$$

then $b^2 > ac$.

The equations to the Curve (D_1) in (κ, λ) are

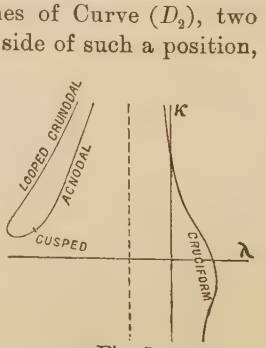


Fig. 7.

implicitly found in § 6. The conic locus of nodes in this sub-group is

$$ax^2-2bxy-3cy^2+ex-3dy=0.$$

In the case considered, this conic is supposed to be an ellipse, that is, (b^2+3ac) is negative, so that x, y, and therefore κ, are limited; there are only two asymptotes parallel to the (κ) axis. The singular quartics are crunodal with loops, acnodal, and cuspidal, if (κ, λ) be a point on the loop of Curve (D_1). The cuspidal quartic corresponds to one of the two limiting values of (x, y) in the locus of nodes, and therefore also of (κ, λ) in Curve (D_1).

The quartics are crunodal and cruciform, which correspond to points (κ, λ) on the serpentine portion of Curve (D_1).

26. To determine the number of points of inflexion of quartics in the hyperbolic division, by the aid of the second discriminating curve—Curve (D_2).

The implicit equations in (κ, λ) to Curve (D_2) are given in § 10.

The number of inflections is 6, 4, or 2, according to the position of (κ, λ) with respect to the several branches of Curve (D_2).

The corresponding quartic has a folium-point, or point of undulation, if (κ, λ) is on a branch.

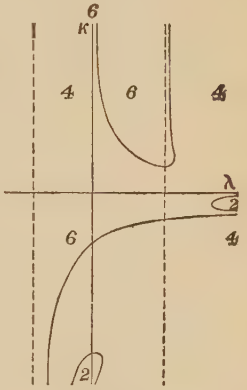


Fig. 8.

27. In the special case, when an asymptote of the satellite hyperbola is parallel to a triple asymptote of the quartic, the cusp at infinity becomes a triple point, and the quartics are of the sixth class. (Dr. Salmon's Genus VII.)

In this case $c=0$, and the quartic sub-group is thus denoted:

$$2\kappa x^3y=ax^2+2bxy+2ex+2dy+\lambda.$$

For a critical value,

$$\kappa x^3=bx+d, \quad ax^2+2ex+\lambda=0.$$

Hence y is given by a vanishing fraction, and a linear equation in x must be a factor of the quartic.

All singular quartics in this special sub-group are resoluble into cubics and lines parallel to the (y) axis.

For all non-singular quartics, the three asymptotes, real or imaginary, are given by the cubic $\kappa x^3=bx+d$.

The equations to Curve (D_1), which is unipartite, are given above implicitly. (Fig. 9.)

28. To determine the number of inflexions of quartics in this special case, by the aid of the second discriminating curve.

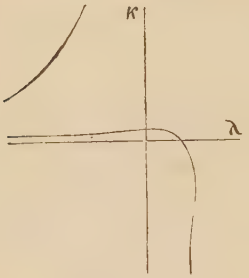


Fig. 9.

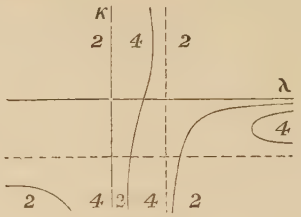


Fig. 10.

The numbers on either side of the branches of Curve (D_2) will make the sequence of inflexions clear, as previously explained.

29. If $c = 0$ and $d = 0$, Curve (D_1) is a bicusped quartic, and Curve (D_2) is modified, and the sequence of inflections altered.

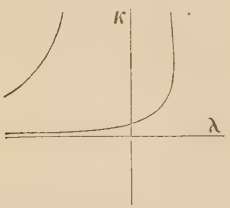


Fig. 11.

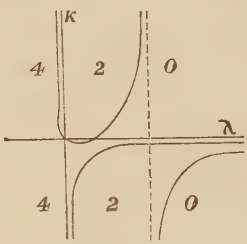


Fig. 12.

If $a = c = d = e = 0$, there is no singular quartic, and no folium-point in any curve of this sub-group.

30. In the special case, when the satellite is resolved into two lines, which are triple asymptotes of the quartic, the cusp is merged in a stapete-point at infinity, and the quartics are of the fourth class.

Let the equation to the sub-group be thus denoted:

$$2\kappa x^3 y = ax^2 + 2ex + \lambda, \quad \text{and} \quad b = c = d = 0 \text{ in § 5.}$$

$$S = \tfrac{1}{12}\, a^2\eta^4 + \kappa\eta^3\,(2\lambda\xi + e),$$

$$T = -\tfrac{1}{2}\tfrac{1}{16}\, a^3\eta^6 - \tfrac{1}{12}\kappa a\eta^5\,(e - 4\lambda\xi) - \tfrac{1}{4}\kappa^2\lambda\eta^4 - \tfrac{1}{2}\kappa e^2\xi\eta^5.$$

For the equivalent class-curve $S^3 = 27T^2$.

The factor η^8 is rejected for the stapete-point, at which two cusps and a node disappear; and a class quartic remains (Salmon's Div. IX.) There is no further singularity; but a line of folium points $(2a\lambda + 3e^2 = 0)$ represents Curve (D_2), and such quartics are denoted $2\kappa a x^3 y = (ax + 3e)\,(ax - e)$.

31. Quartics, which have a cusp at infinity, and the line at infinity for a tangent thereat, may be written

$$(x^3 + 3bx^2)\, y + u_2 = 0.$$

By changing the origin to the point $(-b, 0)$, they may be reduced to

the form considered in this memoir,

$$x^3y + u_2 = 0,$$

in which $(x = 0)$ is the determinate line passing through the cusp and the two points of osculation of the osculating conic.

The general form of quartics, which are simply met by the line at infinity in three coincident points and another point and which have a triple and a single asymptote $(x^3y + u_3 + u_2 = 0)$, is not here considered.

[The assumed form $\kappa\alpha^3\beta = u_2$, or, as this is afterwards written, $2\kappa x^3y = ax^2 + 2bxy + cy^2 + 2ex + 2dy + \lambda$, is, I think, introduced without a proper explanation. Say, the form is $x^3y = z^2\,(*\,\mathfrak{L}x, y, z)^2$, it ought to be shown how for a cuspidal quartic we arrive at this form; viz., taking the cusp to be at the point $(x=0, z=0)$, $z=0$ for the tangent at the cusp, and $x=0$ an arbitrary line through the cusp; then the line $z=0$ besides intersects the curve in a single point, and, if $y=0$ is taken as the tangent at that point, the equation of the curve must, it can be seen, be of the form

$$(x^3 + \theta x^2 z)\, y = z^2\,(a, b, c, f, g, h\mathfrak{L}x, y, z)^2.$$

The conic $(a, b, c, f, g, h\mathfrak{L}x, y, z)^2 = 0$ touches the quartic at each of the two intersections of the quartic with the arbitrary line $x=0$; and we cannot, so long as the line remains arbitrary, find a conic which shall osculate the quartic at the two points in question; but, for the particular line $x + \frac{1}{3}\theta z = 0$, there exists such a conic, viz., writing x instead of $x + \frac{1}{3}\theta z$, the form is $x^3y = z^2\,(a', b', c', f', g', h'\mathfrak{L}x, y, z)^2$, and the new conic $(a', \ldots\mathfrak{L}x, y, z)^2 = 0$ has the property in question. This is the adopted form, and it thus appears that in it the line $x = 0$ is a determinate line, viz., the line passing through the cusp and the two points of osculation of the osculating conic. It thus appears that in the assumed form the lines $x = 0$, $y = 0$, $z = 0$ are determinate lines.—A. C.]

Note on an Exceptional Case in which the Fundamental Postulate of Professor Sylvester's Theory of Tamisage fails. By Mr. J. Hammond, M.A.

[*Read Dec. 14th*, 1882.]

1. The expansion of the generating function for the binary seventhic is

$$1 + ax^7 + a^2\,(x^2 + x^6 + x^{10} + x^{14})$$
$$+ a^3\,(x^3 + x^5 + x^7 + 2x^9 + x^{11} + x^{13} + x^{15} + x^{17} + x^{21})$$
$$+ a^4\,(\ldots + x^6 + \ldots) + a^5\,(\ldots + 4x^{15} + \ldots) + \ldots;$$

whence it is evident that there are four, and only four, compound covariants of degree-order (5 . 13), viz., (1 . 7) (4 . 6), (2 . 2) (3 . 11), (2 . 6) (3 . 7), and (2 . 10) (3 . 3).

If, then, the fundamental postulate is true, there is neither a ground-form nor a syzygant of degree-order (5 . 13) for the binary seventhic.

The object of this short note is to prove that there is both a ground-form and a syzygant of degree-order (5 . 13) for the binary seventhic.

The covariant (1 . 7) is the seventhic itself, viz.,

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7\mathcal{X}x, y)^7;$$

but, if, as is usually done, we represent the covariants by their sources,

$$(1.7) = a_0,$$
$$(2.10) = a_0a_2 - a_1^2,$$
$$(2.6) = a_0a_4 - 4a_1a_3 + 3a_2^2,$$
$$(2.2) = a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2,$$
$$(3.11) = a_0^2a_5 - 5a_0a_1a_4 + 2a_0a_2a_3 + 8a_1^2a_3 - 6a_1a_2^2,$$
$$(3.7) = a_0^2a_7 - 7a_0a_1a_6 + 9a_0a_2a_5 + 12a_1^2a_5 - 5a_0a_3a_4 - 30a_1a_2a_4 + 20a_1a_3^2,$$
$$(3.3) = 2a_0a_2a_7 - 2a_1^2a_7 - 7a_0a_3a_6 + 7a_1a_2a_6 + 5a_0a_4a_5 + 22a_1a_3a_5 - 27a_2^2a_5 - 25a_1a_4^2 + 45a_2a_3a_4 - 20a_3^3.$$

The remaining sources to be considered are all of weight 11; three of them, viz., (2 . 2) (3 . 11), (2 . 6) (3 . 7), and (2 . 10) (3 . 3), are obtained by multiplication from the above; (4 . 6) is found from the differential equation and multiplied by (1 . 7), which does not alter the coefficients; Θ is the ground-source of degree-order (5 . 13) for the seventhic, and is No. 11 in the list of sources for the quintic in Salmon's *Higher Algebra* (3rd ed., p. 216), or it is *O* in Prof. Cayley's "Tables for the Binary Sextic" (*American Journal of Mathematics*, Vol. iv., p. 382). All of these are given in a tabular form to avoid repetition.

	$a_1a_2^2a_3^2$	$a_1^2a_3^3$	$a_0a_2a_3^2$	$a_1a_2^3a_4$	$a_1^2a_2a_3a_4$	$a_0a_2^2a_3a_4$	$a_0a_1a_3^2a_4$
Θ	−2	8	−6	3	−16	14	−1
(2.10)(3.3)		20	−20		−45	45	
(2.6)(3.7)	60	−80		−90	120	−15	40
(2.2)(3.11)	60	−80	−20	−90	120	30	50
(1.7)(4.6)			20			−45	−10

	$a_1^3 a_4^2$	$a_0 a_1 a_2 a_4^2$	$a_0^2 a_3 a_4^2$	$a_1^2 a_2^2 a_5$	$a_0 a_2^3 a_5$	$a_1^3 a_3 a_5$	$a_0 a_1 a_2 a_3 a_5$	$a_0^2 a_3^2 a_5$
Θ	9	−11	2	6	−9	−8	14	−3
&c.	25	−25		27	−27	−22	22	
		−30	−5	36	27	−48	−36	
		−75		36		−48	−12	−10
		45	−5		27		−24	10

	$a_0 a_1^2 a_4 a_5$	$a_0^2 a_2 a_4 a_5$	$a_0^2 a_1 a_5^2$	$a_1^3 a_2 a_6$	$a_0 a_1 a_2^2 a_6$	$a_0 a_1^2 a_3 a_6$	$a_0^2 a_2 a_3 a_6$
Θ	−1	1					
&c.	−5	5		−7	7	7	−7
	12	9			−21	28	
	30	15	−6		−6	8	2
	−18	−6	6		−15	20	−2

	$a_0^2 a_1 a_4 a_6$	$a_0^3 a_5 a_6$	$a_1^4 a_7$	$a_0 a_1^2 a_2 a_7$	$a_0^2 a_2^2 a_7$	$a_0^2 a_1 a_3 a_7$	$a_0^3 a_4 a_7$
(2.10)(3.3)			2	−4	2		
(2.6)(3.7)	−7				3	−4	1
(2.2)(3.11)	−5	1					
(1.7)(4.6)	−2	−1			3	−4	1

An inspection of the five tabulated sources will show that

$$(2.6)(3.7)-(2.2)(3.11)=(1.7)(4.6),$$

and that it is impossible to express Θ in the form

$$\Theta = p(2.10)(3.3)+q(2.6)(3.7)+r(2.2)(3.11)+s(1.7)(4.6).$$

In other words, the covariant Θ cannot be expressed as a rational integral function of ground-forms of lower degree-order; or there is a ground-form, Θ, and a syzygant,

$$(1.7)(4.6)+(2.2)(3.11)-(2.6)(3.7),$$

both of degree-order (5.13), for the binary seventhic.

2. Let Q_{2m} denote the quadrinvariant of the $2m$-ic and let C_{2m+1} be the source of a covariant of degree 3 in the coefficients, such that

$$C_{2m+1} = a_0 V_{2m+1} - 2a_1 Q_{2m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

then, if $$\delta = a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \ldots,$$

operating with δ on (1), and remembering that $\delta Q_{2m} = 0$, $\delta C_{2m+1} = 0$, we have $$\delta V_{2m+1} = 2Q_{2m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) it follows that

$$C_{2m+1} Q_{2p} - Q_{2m} C_{2p+1} = a_0 (V_{2m+1} Q_{2p} - V_{2p+1} Q_{2m}),$$

where $V_{2m+1} Q_{2p} - V_{2p+1} Q_{2m}$ is a source of degree-weight $(4.2m+2p+1)$.

In the case before us $m = 3$, $p = 2$, $2m+2p+1 = 11$, and

$$V_{2m+1} Q_{2p} - V_{2p+1} Q_{2m}$$

is the source of a covariant of degree-order (4.6) for the binary seventhic.

Replacing the sources by the corresponding covariants, we have

$$(3.7)(2.6) - (2.2)(3.11) = (1.7)(4.6) \quad \ldots\ldots\ldots\ldots (3),$$

a syzygant of degree-order (5.13).

Now, since there are four linearly independent covariants of this degree-order, and four compound covariants, of which (3) shows that there are only three linearly independent; it follows that there must be a ground-form of this degree-order to make up the number of independent covariants.

The source of this ground-form is Θ, which is known to be a ground-source [of degree-order $(5.3)(5.8)$] for the quintic and sextic.

It may be shown that Θ is *not a ground-source* for any quantic of higher order than the seventhic.

Addition to the foregoing Paper. *By* Prof. Cayley.

[*Read Dec. 14th*, 1882.]

The extreme importance of Mr. Hammond's result, as regards the entire subject of Covariants, leads me to reproduce his investigation in the notation of my Memoirs on Quantics. and with a somewhat different arrangement of the formulæ. For the binary seventhic

$$(a, b, c, d, e, f, g, h\,\mathcal{X}\,x, y)^7,$$

the four composite seminvariants of the deg-weight 5.11 (sources of covariants of the deg-order 5.13) are

I.			II.		
1.7	4.6	... ...	2.10	3.3	Deg-order.
1.0	4.11	... ...	2.2	3.9	Deg-weight.
a +1	a^2eh +1		ac +1	ach +2	
	fg −1		b^2 −1	dg −7	
	$abdh$ −4			ef +5	
	beg −2			a^0b^2h −2	
	bf^2 +6			bcg +7	
	c^2h +3			bdf +22	
	cdg −2			be^2 −25	
	cef −6			c^2f −27	
	d^2f +10			cde +45	
	de^2 −5			d^3 −20	
	a^0b^3ch 0				
	b^2dg +20				
	b^2ef +57				
	bc^2g −15				
	$bcdf$ −24				
	bce^2 −30				
	bd^2e −10				
	c^3f +27				
	c^2de −45				
	cd^3 +20				

III.			IV.		
2.6	3.7	... ...	2.2	3.11	Deg-order.
2.4	3.7	... ...	2.6	3.5	Deg-weight.
ae +1	a^2h +1		ag +1	a^2f +1	
bd −4	abg −7		bf −6	abe −5	
c^2 +3	cf +9		ce +15	cd +2	
	de −5		d^2 −10	a^0b^2d +8	
	a^0b^2f +12			bc^2 −6	
	bce −30				
	bd^2 +20				

and it is here at once obvious that there exists a syzygy of the form I. = III. − IV. ; in fact, if in III. and IV. we write $a = 0$, then the values are each

$$= -2b\,(4bd - 3c^2)\,(6bf - 15ce + 10d^2)\,;$$

hence III.—IV. must divide by a, the quotient being a seminvariant of the deg-weight 4 . 11, which can only be a numerical multiple of the second factor of I., and is in fact = this second factor, that is, we have the syzygy I. = III.—IV.

Working out the values of the four products, and joining to them the expression for the irreducible seminvariant of the same deg-weight 5 . 11 (O, x^8 of my tables for the binary sextic), we have the table:

5 . 10
a^3dh
eg
f
a^2bch
bdg
bef
c^2g
cdf
ce^2
d^2e
ab^3h
b^2cg
b^2df
b^2e^2
bc^2f
$bcde$
bd^3
c^3e
c^2d^2
a^0b^4g
b^3cf
b^3de
b^2c^2e
b^2cd^2
bc^3d
c^5

5 . 11	O	I.	III.	IV.	II.
a^3eh		+1	+1		
fg		−1		+1	
a^2bdh		−4	−4		
beg		−2	−7	−5	
bf^2		+6		−6	
c^2h		+3	+3		+2
cdg		−2		+2	−7
cef	−1	−6	+9	+15	+5
d^2f	+3	+10		−10	
de^2	−2	−5	−5		
ab^2ch					−4
b^2dg		+20	+28	+8	+7
b^2ef	+1	+57	+12	−45	−5
bc^2g		−15	−21	−6	+7
$bcdf$	−14	−24	−36	−12	+22
bce^2	+11	−30	−30		−25
bd^2e	+1	−10	+40	+50	
c^3f	+9	+27	+27		−27
c^3de	−14	−45	−15	+30	+45
cd^3	+6	+20		−20	−20
a^0b^4h					−2
b^3cg					−7
b^3df	+8		−48	−48	−22
b^3e^2	−9				+25
b^2c^2f	−6		+36	+36	+27
b^2cde	+16		+120	+120	−45
b^2d^3	−8		−80	−80	+20
bc^3e	−3		−90	−90	
bc^2d^2	+2		+60	+60	
c^4d					

I have prefixed to the table the literal terms of the deg-weight 5 . 10; for the deg-weights 5 . 11 and 5 . 10 the numbers of terms are = 30 and 26 respectively; and it is the difference of these 30 − 26, = 4, which gives the number of asyzygetic seminvariants of the deg-weight 5 . 11.

On Compound Determinants. *By* R. F. Scott.

[*Read Nov. 9th*, 1882.]

1. Consider the determinant of order $n+m$,

$$D = \begin{vmatrix} a_{11}, & a_{12}, & \dots & a_{1n}, & b_{11}, & \dots & b_{1m} \\ a_{21}, & a_{22}, & \dots & a_{2n}, & b_{21}, & \dots & b_{2m} \\ \dots & \dots & & \dots & \dots & & \\ a_{n1}, & a_{n2}, & \dots & a_{nn}, & b_{n1}, & \dots & b_{nm} \\ c_{11}, & c_{12}, & \dots & c_{1n}, & h_{11}, & \dots & h_{1m} \\ \dots & \dots & & \dots & \dots & & \\ c_{m1}, & c_{m2}, & \dots & c_{mn}, & h_{m1}, & \dots & h_{mm} \end{vmatrix},$$

which we may regard as made up of four blocks of elements. A block of n rows and columns of elements a_{ik}; a block of n rows and m columns of elements b_{ik}; a block of m rows and n columns of elements c_{ik}; and finally, a block of m rows and columns of elements h_{ik}.

Let A stand for the determinant of order n,

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix},$$

and let the system of first minors of the determinant A be denoted by

$$\begin{vmatrix} A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots \\ A_{n1} & \dots & A_{nn} \end{vmatrix}.$$

If we border the block of elements a_{ik} in D with a row and column from the blocks of elements c_{ik} and b_{ik}, and that element from the block of elements h_{ik} where these rows and columns intersect, we get a system of m^2 elements of the form

$$p_{ik} = \begin{vmatrix} a_{11} & \dots & a_{1n}, & b_{1k} \\ a_{21} & \dots & a_{2n}, & b_{2k} \\ \dots & \dots & \dots & \\ a_{n1} & \dots & a_{nn}, & b_{nk} \\ c_{i1} & \dots & c_{in}, & h_{ik} \end{vmatrix}.$$

I propose, in the first place, to evaluate the determinant of order m formed by the system of elements p_{ik}.

Observe that, by Cauchy's theorem,

$$p_{ik} = Ah_{ik} - \Sigma b_{rk} c_{is} A_{rs} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Multiply the determinant D by the determinant of order $n+m$,

$$A^{n-1} = \begin{vmatrix} A_{11} & \ldots & A_{1n}, & 0 & \ldots & 0 \\ A_{21} & \ldots & A_{2n}, & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & & \\ A_{n1} & \ldots & A_{nn}, & 0 & \ldots & 0 \\ 0 & \ldots & 0, & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & & \\ 0 & \ldots & 0, & 0 & \ldots & 1 \end{vmatrix}.$$

Then, if we write

$$a_{ik} = c_{i1} A_{k1} + c_{i2} A_{k2} + \ldots + c_{in} A_{kn},$$

we have

$$A^{n-1} D = \begin{vmatrix} A, & 0 & \ldots & 0, & b_{11} & \ldots & b_{1m} \\ 0, & A & \ldots & 0, & b_{21} & \ldots & b_{2m} \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0, & 0 & \ldots & A, & b_{n1} & \ldots & b_{nm} \\ a_{11}, & a_{12} & \ldots & a_{1n}, & h_{11} & \ldots & h_{1m} \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ a_{m1}, & a_{m2} & \ldots & a_{mn}, & h_{m1}, & \ldots & h_{mm} \end{vmatrix}.$$

Now divide the first n rows of the determinant on the right by A, and then multiply the last m columns by A; this divides the whole determinant by A^{n-m}, and we have therefore

$$A^{m-1} D = \begin{vmatrix} 1, & 0 & \ldots & 0, & b_{11} & \ldots & b_{1m} \\ 0, & 1 & \ldots & 0, & b_{21} & \ldots & b_{2m} \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0, & 0 & \ldots & 1, & b_{n1} & \ldots & b_{nm} \\ a_{11}, & a_{12} & \ldots & a_{1n}, & Ah_{11} & \ldots & Ah_{1m} \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ a_{m1}, & a_{m2} & \ldots & a_{mn}, & Ah_{m1} & \ldots & Ah_{mm} \end{vmatrix}.$$

Multiply the first n rows of this determinant by

$$a_{i1}, \quad a_{i2} \quad \ldots \quad a_{in}$$

respectively, and subtract their sum from the $(n+i)^{\text{th}}$ row; do this for the values $i = 1, 2 \ldots n$. Then the element at the intersection of the $(n+i)^{\text{th}}$ row and $(n+k)^{\text{th}}$ column will be

$$\begin{aligned} & Ah_{ik} - (a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{in} b_{nk}) \\ &= Ah_{ik} - \Sigma b_{rk} c_{is} A_{rs} \\ &= p_{ik} \text{ by (1).} \end{aligned}$$

Hence, we have

$$A^{m-1}D = \begin{vmatrix} 1, & 0 \ldots & 0, & b_{11} & \ldots & b_{1m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0, & 0 \ldots & 1, & b_{m1} & \ldots & b_{nm} \\ 0, & 0 \ldots & 0, & p_{11} & \ldots & p_{1m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0, & 0 \ldots & 0, & p_{m1} & \ldots & p_{mm} \end{vmatrix}$$

Or

$$A^{m-1}D = \begin{vmatrix} p_{11} & \ldots & p_{1m} \\ \ldots & \ldots & \ldots \\ p_{m1} & \ldots & p_{mm} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

That is to say, we have evaluated the determinant in question.

2. Next, let us border the block of elements a_{ik} with r rows from the block of elements c_{ik} and with r columns from the block of elements b_{ik}, completing the determinant with those elements from the block of elements h_{ik} which are situated at the intersection of these last rows and columns. We should thus get

$$\left\{ \frac{m\,(m-1)\,\ldots\,(m-r+1)}{1\,.\,2\,\ldots\,r} \right\}^2 = \{m_r\}^2 = \nu^2,$$

elements which we may denote by

$$\begin{matrix} q_{11} & \ldots & q_{1\nu} \\ \ldots & \ldots & \ldots \\ q_{\nu 1} & \ldots & q_{\nu\nu} \end{matrix}$$

I propose to find the value of the determinant q formed from these elements.

Consider that determinant of the system which is obtained by taking the first r rows and columns from the blocks of elements c_{ik} and b_{ik}. We may suppose that this is q_{11}; thus

$$q_{11} = \begin{vmatrix} a_{11} & \ldots & a_{1n}, & b_{11} & \ldots & b_{1r} \\ \ldots & \ldots & \ldots & \ldots & & \\ a_{n1} & \ldots & a_{nn}, & b_{n1} & \ldots & b_{nr} \\ c_{11} & \ldots & c_{1n}, & h_{11} & \ldots & h_{1r} \\ \ldots & \ldots & \ldots & \ldots & & \\ c_{r1} & \ldots & c_{rn}, & h_{r1} & \ldots & h_{rr} \end{vmatrix}.$$

But it is clear, from equation (2) of §1, that

$$A^{r-1}q_{11} = \begin{vmatrix} p_{11} & \ldots & p_{1r} \\ \ldots & \ldots & \ldots \\ p_{r1} & \ldots & p_{rr} \end{vmatrix}.$$

That is to say, q_{11} when multiplied by A^{r-1} is a minor of order r of the determinant

$$p = |\,p_{ik}\,|\,.$$

If we remember that all the elements in the i^{th} row of p are obtained by bordering the system a_{ik} with the row of elements $c_{i1}, c_{i2} \ldots c_{in}$, and all the elements in the k^{th} column of p by bordering the system a_{ik} with the column of elements $b_{1k}, b_{2k} \ldots b_{nk}$, we see in the same way that each element of the system q_{ik} when multiplied by A^{r-1} is a minor of the determinant p. Hence the determinant

$$\begin{vmatrix} A^{r-1}q_{11} & \ldots & A^{r-1}q_{1r} \\ \ldots & \ldots & \ldots \\ A^{r-1}q_{r1} & \ldots & A^{r-1}q_{rr} \end{vmatrix}$$

is equal to the determinant formed with all the minors of order r of the determinant p. That is to say, it is equal to

$$p^{(m-1)_{r-1}}.$$

Now substitute for p its value from equation (2), and we see that

$$q = A^{(m-1)_r} D^{(m-1)_{r-1}}.$$

For, on dividing out the common factors on both sides of the equation, the exponent of A is

$$\begin{aligned}(m-1)(m-1)_{r-1} - (r-1)\, m_r &= m_r - (m-1)_{r-1} \\ &= (m-1)_r.\end{aligned}$$

3. The theorems of §§ 1 and 2 were given, without proof, by Professor Sylvester in the *Philosophical Magazine* for 1851, in his paper "On the Relation between the Minor Determinants of Linearly Equivalent Quadratic Forms"; a misprint in the result being corrected by the learned author in his note "On a Theorem concerning the Combination of Determinants." (*Camb. and Dublin Math. Jour.*, viii., 60.) The only general proof of them which has been published, so far as I am aware, is that given by M. Picquet in his "Mémoire sur l'application du Calcul des Combinaisons à la Théorie des Déterminants." (*Journal de l'Ecole Pol.*, xxviii., *Théorème IX.*) This proof is of an inductive character, and not, perhaps, very easy to follow.* The theorem of §2 has been given again by Professor Sylvester in his paper "Sur les Déterminants Composés," *Crelle*, lxxxviii., under the form

$$B_p^i \,.\, A_n = B^\sigma \,.\, (AB)^\tau,$$

where $$\sigma = (n-1)_i, \quad \tau = (n-1)_{i-1},$$

B and (AB) being the determinants here called A and D respectively, while i and n are written for r and m respectively. $B_p^i \,.\, A_n$ is Professor Sylvester's symbol for the determinant q. The theorem is there also stated without proof.

---

* Since the above paper was read before the Society, Mr. Muir has drawn my attention to a proof of these theorems given by Reiss,—*Beiträge zur Theorie der Determinanten* (Leipzig, 1867), p. 34.

4. Partly for convenience of notation and partly on account of what follows, I have divided up the determinant D into four blocks of elements denoted by different letters. But we may enunciate the theorem of § 2 as follows:—Having given a minor A of order n, of a determinant D of order $n+m$, if we border the minor in all possible ways with r of the remaining rows and columns, we get the elements of a new determinant q of order m_r, whose value is

$$A^{(m-1)_r} D^{(m-1)_{r-1}}.$$

And this (allowing for the difference of notation), is the enunciation of M. Picquet's Théorème IX. The theorem of § 1 is the special case of this when $r = 1$.

5. In what follows, we shall have occasion to consider certain determinants arising from the elements of three given determinants and their minors. I shall denote these determinants by

$$A = |\, a_{ik} \,|, \quad B = |\, b_{ik} \,|, \quad C = |\, c_{ik} \,| \quad (i, k = 1, 2 \ldots n).$$

The determinants formed from the systems of first minors of these determinants will be denoted by

$$|\, A_{ik} \,|, \quad |\, B_{ik} \,|, \quad |\, C_{ik} \,|.$$

Let $$\frac{n(n-1)\ldots(n-m+1)}{1 \cdot 2 \ldots m} = m_m = \mu,$$

and let $$|\, a_{ik} \,| \quad (i, k = 1, 2 \ldots \mu)$$

denote the system of minors of the determinant A of order m.

Let $$|\, a'_{ik} \,|$$

denote the system of minors of order $n-m$ of the determinant A, so that a_{ik} and a'_{ik} are complementary.

Similarly, let β_{ik}, β'_{ik}; γ_{ik}, γ'_{ik} denote the like systems of minors for the determinants B and C. It will be supposed throughout that the selection of rows and columns correspond in all these compound systems; so that, for instance, we obtain β_{ik} from a_{ik} by simply substituting b for a throughout that minor. Also the system of elements $a_{i1}, a_{i2} \ldots a_{i\mu}$ is obtained by taking the i^{th} combination of m rows of A and forming all possible determinants of order m from its elements. The minor a_{ik} is common to the i^{th} selection of rows and the k^{th} selection of columns.

As we shall frequently have to refer to certain known theorems concerning minors of these systems, it will be convenient for purposes of reference to state them here.

We have then

$$\begin{vmatrix} a_{11} & \dots & a_{1r} \\ \dots & \dots & \dots \\ a_{r1} & \dots & a_{rr} \end{vmatrix} = A^{r-(n-1)m} \begin{vmatrix} a'_{r+1\,r+1} & \dots & a'_{r+1\,\mu} \\ \dots & \dots & \dots \\ a'_{\mu\,r+1} & \dots & a'_{\mu\mu} \end{vmatrix} \quad \dots\dots\dots (1),$$

$$\begin{vmatrix} a'_{11} & \dots & a'_{1r} \\ \dots & \dots & \dots \\ a'_{r1} & \dots & a'_{rr} \end{vmatrix} = A^{r-(n-1)m-1} \begin{vmatrix} a_{r+1\,r+1} & \dots & a_{r+1\,\mu} \\ \dots & \dots & \dots \\ a_{\mu\,r+1} & \dots & a_{\mu\mu} \end{vmatrix} \quad \dots\dots\dots (2),$$

$$A^{(n-1)m-1^{n-1}} a'_{rs} = \frac{d}{a_{rs}} \{ | a_{ik} | \} \quad \dots\dots\dots (3).$$

Here and elsewhere, in the present paper, instead of taking the general minor of any order r, I take that common to the first r rows and columns. There is clearly no loss of generality in this, as, by a suitable interchange of rows and columns, any selected r rows and columns can be brought into the first place. The simplification in the notation required is very considerable.

6. If we border the determinant A with a column from B and a row from C, we get a new system of n^2 elements given by

$$p_{ik} = \begin{vmatrix} a_{11} & \dots & a_{1n}, & b_{1k} \\ a_{21} & \dots & a_{2n}, & b_{2k} \\ \dots & \dots & \dots & \\ a_{n1} & \dots & a_{nn}, & b_{nk} \\ c_{i1} & \dots & c_{in}, & 0 \end{vmatrix}.$$

Let p denote the determinant of this system, so that

$$p = \begin{vmatrix} p_{11} & \dots & p_{1n} \\ \dots & \dots & \dots \\ p_{n1} & \dots & p_{nn} \end{vmatrix}.$$

Then it is clear, from equation (2) of § 1, that

$$p = A^{n-1} \begin{vmatrix} a_{11} & \dots & a_{1n}, & b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots & & \\ a_{n1} & \dots & a_{nn}, & b_{n1} & \dots & b_{nn} \\ c_{11} & \dots & c_{1n}, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & & \\ c_{n1} & \dots & c_{nn}, & 0 & \dots & 0 \end{vmatrix}$$

$$= (-1)^n A^{n-1} BC.*$$

---

* Another proof of this theorem is given in *The Messenger of Mathematics*, xi., p. 96. See also Muir, "On the Multiplication of the $(n-1)^{th}$ Power, &c." (*American Journal of Mathematics*, iv.)

Observe, also, that the k^{th} column of the determinant p corresponds to the k^{th} column of the determinant B, and the i^{th} row of p to the i^{th} row of C. Thus, by the same theorem of § 1, we have*

$$\begin{vmatrix} p_{11} & \dots & p_{1r} \\ \dots & \dots & \dots \\ p_{r1} & \dots & p_{rr} \end{vmatrix} = A^{r-1} \begin{vmatrix} a_{11} & \dots & a_{1n}, & b_{11} & \dots & b_{1r} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn}, & b_{n1} & \dots & b_{nr} \\ c_{11} & \dots & c_{1n}, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{r1} & \dots & c_{rn}, & 0 & \dots & 0 \end{vmatrix}.$$

Similarly, any minor of p of order m, say π_{ik}, is equal to the determinant obtained by bordering the determinant A with the k^{th} selection of the columns of B and the i^{th} selection of the rows of C, multiplied by A^{m-1}.

7. Let us take the determinant

$$f_{11} = \begin{vmatrix} a_{11} & \dots & a_{1n}, & b_{11} & \dots & b_{1m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn}, & b_{n1} & \dots & b_{nm} \\ c_{11} & \dots & c_{1n}, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{m1} & \dots & c_{mn}, & 0 & \dots & 0 \end{vmatrix},$$

and expand it according to products of minors of order m from the last m rows and columns. Then, assuming that the first m rows and columns go to form the first selection in § 5, we have

$$f_{11} = (-1)^m \Sigma a'_{rs} \beta_{r1} \gamma_{1s} \dots\dots\dots\dots\dots\dots\dots\dots (3).$$

Now, substitute for a'_{rs} its value from equation (3), § 5, and replace the sum on the right of (3) by the equivalent determinant from Cauchy's theorem, and we have

$$f_{11} A^{(n-1)_{m-1}-1} = (-1)^{m-1} \begin{vmatrix} a_{11} & \dots & a_{1\mu}, & \beta_{11} \\ \dots & \dots & \dots & \dots \\ a_{\mu 1} & \dots & a_{\mu\mu}, & \beta_{\mu 1} \\ \gamma_{11} & \dots & \gamma_{1\mu}, & 0 \end{vmatrix}.$$

In exactly the same way, if f_{ik} stands for the determinant obtained by bordering the determinant A with the k^{th} selection of the columns of B and the i^{th} selection of the rows of C, we should have

$$f_{ik} A^{(n-1)_{m-1}-1} = (-1)^{m-1} \begin{vmatrix} a_{11} & \dots & a_{1\mu}, & \beta_{1k} \\ \dots & \dots & \dots & \dots \\ a_{\mu 1} & \dots & a_{\mu\mu}, & \beta_{\mu k} \\ \gamma_{i1} & \dots & \gamma_{i\mu}, & 0 \end{vmatrix} \dots\dots\dots (4).$$

* Cf M. C. le Paige, "Note sur les déterminants bordés," *Bulletin de la Soc. Math. de France*, viii., 128.

8. If we border the determinant A in all possible ways with m columns from the determinant B, and m rows from the determinant C, we get a system of μ^2 elements

$$\begin{matrix} f_{11} & \cdots & f_{1\mu} \\ \cdots & \cdots & \cdots \\ f_{\mu 1} & \cdots & f_{\mu\mu}. \end{matrix}$$

In connection with this, let us consider the conjugate system

$$\begin{matrix} g_{11} & \cdots & g_{1\mu} \\ \cdots & \cdots & \cdots \\ g_{\mu 1} & \cdots & g_{\mu\mu}, \end{matrix}$$

where g_{ik} is obtained by bordering the determinant A with those $n-m$ columns of B and $n-m$ rows of C complementary to the columns and rows taken to form f_{ik} by bordering the determinant A.

In the first place, let us consider the determinant $|f_{ik}|$. Substitute for f_{ik} its value from equation (4), § 7, and reduce the determinant on the right by the first theorem of § 6. Thus

$$|f_{ik}| A^x (-1)^{m\mu-\mu} = (-1)^\mu \{ |\alpha_{ik}| \}^{\mu-1} . |\beta_{ik}| . |\gamma_{ik}|,$$

where $$x = \mu\{(n-1)_{m-1}-1\}.$$

Dividing out the common powers of A, the exponent of A on the right becomes

$$-\mu\{(n-1)_{m-1}-1\}+(\mu-1)(n-1)_{m-1}$$

$$= \mu-(n-1)_{m-1} = (n-1)_m.$$

Hence $$|f_{ik}| = (-1)^{m\mu} A^{(n-1)_m} (BC)^{(n-1)_{m-1}}.$$

Similarly we should find*

$$|g_{ik}| = (-1)^{(n-m)\mu} A^{(n-1)_{m-1}} (BC)^{(n-1)_m}.$$

9. The ratio of complementary minors of the systems f_{ik} and g_{ik} is equal to a power of A multiplied by a power of BC.

In the minor $$\begin{vmatrix} f_{11} & \cdots & f_{1p} \\ \cdots & \cdots & \cdots \\ f_{p1} & \cdots & f_{pp} \end{vmatrix}.$$

Substitute for f_{ik} its value from equation (4) of § 7. Then, if we write $$q = (n-1)_{m-1}-1,$$

---

* Cf. *Messenger of Mathematics*, xi., 98.

we have, by the theorem of § 1,

$$(-1)^{p(m-1)} A^{pq} \begin{vmatrix} f_{11} & \dots & f_{1p} \\ \dots & \dots & \dots \\ f_{p1} & \dots & f_{pp} \end{vmatrix} = \{ \mid a_{ik} \mid \}^{p-1} \begin{vmatrix} a_{11} & \dots & a_{1\mu}, & \beta_{11} & \dots & \beta_{1p} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{\mu 1} & \dots & a_{\mu\mu}, & \beta_{\mu 1} & \dots & \beta_{\mu p} \\ \gamma_{11} & \dots & \gamma_{1\mu}, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{p1} & \dots & \gamma_{p\mu}, & 0 & \dots & 0 \end{vmatrix}$$

..... (5).

Now let us expand the determinant

$$\begin{vmatrix} a_{11} & \dots & a_{1\mu}, & \beta_{11} & \dots & \beta_{1p} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{\mu 1} & \dots & a_{\mu\mu}, & \beta_{\mu p} & \dots & \beta_{\mu p} \\ \gamma_{11} & \dots & \gamma_{1\mu}, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{p1} & \dots & \gamma_{p\mu}, & 0 & \dots & 0 \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots \quad (6)$$

according to products of minors from the last rows and columns.

One term of the sum is

$$(-1)^p \begin{vmatrix} \beta_{11} & \dots & \beta_{1p} \\ \dots & \dots & \dots \\ \beta_{p1} & \dots & \beta_{pp} \end{vmatrix} \begin{vmatrix} \gamma_{11} & \dots & \gamma_{1p} \\ \dots & \dots & \dots \\ \gamma_{p1} & \dots & \gamma_{pp} \end{vmatrix} \begin{vmatrix} a_{p+1\,p+1} & \dots & a_{p+1\,\mu} \\ \dots & \dots & \dots \\ a_{\mu\,p+1} & \dots & a_{\mu\mu} \end{vmatrix}.$$

Now substitute for these three determinants their values from equations (1) and (2) of § 5, and this term becomes equal to

$$\begin{vmatrix} \beta'_{p+1\,p+1} & \dots & \beta'_{p+1\,\mu} \\ \dots & \dots & \dots \\ \beta'_{\mu\,p+1} & \dots & \beta'_{\mu\mu} \end{vmatrix} \begin{vmatrix} \gamma'_{p+1\,p+1} & \dots & \gamma'_{p+1\,\mu} \\ \dots & \dots & \dots \\ \gamma'_{\mu\,p+1} & \dots & \gamma'_{\mu\mu} \end{vmatrix} \begin{vmatrix} a'_{11} & \dots & a'_{1p} \\ \dots & \dots & \dots \\ a'_{p1} & \dots & a'_{pp} \end{vmatrix}$$

multiplied by $(-1)^p A^{(n-1)m-1-p} (BC)^{p-(n-1)m}$.

The three determinant factors are clearly a term in the expansion of the determinant

$$(-1)^{\mu-p} \begin{vmatrix} a'_{11} & \dots & a'_{1\mu}, & \beta'_{1\,p+1} & \dots & \beta'_{1\mu} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a'_{\mu 1} & \dots & a'_{\mu\mu}, & \beta'_{\mu\,p+1} & \dots & \beta'_{\mu\mu} \\ \gamma'_{p+11} & \dots & \gamma'_{p+1\,\mu}, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma'_{\mu 1} & \dots & \gamma'_{\mu\mu}, & 0 & \dots & 0 \end{vmatrix} \quad \dots\dots\dots\dots \quad (7),$$

according to products of minors from the last $\mu - p$ rows and columns.

And similarly every other term in the expansion of the determinant (6) can be made by a similar transformation to give a term in the expansion of the determinant (7). Hence the ratio of the

determinants

$$\begin{vmatrix} a_{11} & \dots & a_{1\mu}, & \beta_{11} & \dots & \beta_{1p} \\ \dots & \dots & \dots & \dots & & \\ a_{\mu 1} & \dots & a_{\mu\mu}, & \beta_{\mu 1} & \dots & \beta_{\mu p} \\ \gamma_{11} & \dots & \gamma_{1\mu}, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & & \\ \gamma_{p1} & \dots & \gamma_{p\mu}, & 0 & \dots & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a'_{11} & \dots & a'_{1\mu}, & \beta'_{1\,p+1} & \dots & \beta'_{1\mu} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a'_{\mu 1} & \dots & a'_{\mu\mu}, & \beta'_{\mu\,p+1} & \dots & \beta'_{\mu\mu} \\ \gamma'_{p+1\,1} & \dots & \gamma'_{p+1\,\mu}, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma'_{\mu 1} & \dots & \gamma'_{\mu\mu}, & 0 & \dots & 0 \end{vmatrix}$$

is $(-1)^{\mu} A^{(n-1)_{m-1}-p} (BC)^{p-(n-1)_m}$.

But now, by the application of the theorems of §§ 1 and 7, we see that, since $g_{ik} A^{(n-1)_{m-1}} = (-1)^{(n-m-1)} \begin{vmatrix} a'_{11} & \dots & a'_{1\mu}, & \beta'_{1k} \\ \dots & \dots & \dots & \\ a'_{\mu 1} & \dots & a'_{\mu\mu}, & \beta'_{\mu k} \\ \gamma'_{i1} & \dots & \gamma'_{i\mu}, & 0 \end{vmatrix}$,

we have $\{ \mid a'_{ik} \mid \}^{\mu-p-1} \begin{vmatrix} a'_{11} & \dots & a'_{1\mu}, & \beta'_{1\,p+1} & \dots & \beta'_{1\mu} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a'_{\mu 1} & \dots & a'_{\mu\mu}, & \beta'_{\mu\,p+1} & \dots & \beta'_{\mu\mu} \\ \gamma'_{p+1\,1} & \dots & \gamma'_{p+1\,\mu}, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma'_{\mu 1} & \dots & \gamma'_{\mu\mu}, & 0 & \dots & 0 \end{vmatrix}$

$$= \begin{vmatrix} g_{p+1\,p+1} & \dots & g_{p+1\,\mu} \\ \dots & \dots & \dots \\ g_{\mu 1} & \dots & g_{\mu\mu} \end{vmatrix} (-1)^{(n-m-1)(\mu-p)} A^{(\mu-p)[(n-1)_m-1]}.$$

Hence $\begin{vmatrix} f_{11} & \dots & f_{1p} \\ \dots & \dots & \dots \\ f_{p1} & \dots & f_{pp} \end{vmatrix} = (-1)^x A^y (BC)^z \begin{vmatrix} g_{p+1\,p+1} & \dots & g_{p+1\,\mu} \\ \dots & \dots & \dots \\ g_{\mu\,p+1} & \dots & g_{\mu\mu} \end{vmatrix}$,

where

$$x = n\mu - np - m\mu = -nz,$$
$$y = p - (n-1)_{m-1},$$
$$z = p - (n-1)_m,$$

which proves the theorem in question.

10. The theorem of the preceding article can be considered from another point of view; namely, each determinant f_{ik} is, up to a factor, a minor of order m of the determinant p of § 6. For example, by that paragraph, $A^{m-1} f_{11} = \begin{vmatrix} p_{11} & \dots & p_{1m} \\ \dots & \dots & \dots \\ p_{m1} & \dots & p_{mm} \end{vmatrix}$.

And generally, if

$$\begin{matrix} \pi_{11} & \dots & \pi_{1\mu} \\ \dots & \dots & \dots \\ \pi_{\mu 1} & \dots & \pi_{\mu\mu} \end{matrix}$$

enotes the system of minors of order m of the determinant p, and

$$\begin{matrix} \pi'_{11} & \dots & \pi'_{1\mu} \\ \dots & \dots & \dots \\ \pi'_{\mu 1} & \dots & \pi_{\mu\mu} \end{matrix}$$

the corresponding system of minors of order $n-m$, we should have

$$A^{m-1} f_{ik} = \pi_{ik},$$
$$A^{n-m-1} g_{ik} = \pi'_{ik}.$$

Then the theorem of the last paragraph is obtained by writing down the equation (1) of § 5, and substituting for π_{ik} and π'_{ik} in terms of f_{ik} and g_{ik}.

11. By combining §§ 7 and 10, we see that

$$\pi_{ik} A^{(n-1)_{m-1}+m-2} = (-1)^{m-1} \begin{vmatrix} a_{11} & \dots & a_{1\mu}, & \beta_{1k} \\ \dots & \dots & \dots & \\ a_{\mu 1} & \dots & a_{\mu\mu}, & \beta_{\mu k} \\ \gamma_{i1} & \dots & \gamma_{i\mu}, & 0 \end{vmatrix}.$$

12. If we make $m = 1$ in § 9, we get a particular system which is not without interest. In this case, the system f_{ik} becomes identical with that denoted by p_{ik} in § 6, while

$$g_{ik} = \begin{vmatrix} a_{11} & \dots & a_{1n}, & b_{11} & \dots & b_{1\,k-1}, & b_{1\,k+1} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ a_{n1} & \dots & a_{nn}, & b_{n1} & \dots & b_{n\,k-1}, & b_{n\,k+1} & \dots & b_{nn} \\ c_{11} & \dots & c_{1n}, & 0 & \dots & 0, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ c_{i-11} & \dots & c_{i-1n}, & 0 & \dots & 0, & 0 & \dots & 0 \\ c_{i+11} & \dots & c_{i+1n}, & 0 & \dots & 0, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ c_{n1} & \dots & c_{nn}, & 0 & \dots & 0, & 0 & \dots & 0 \end{vmatrix}.$$

Now expand the determinant on the right according to the product of minors from the last $n-1$ rows and columns, thus

$$g_{ik} = (-1)^{n-1} \Sigma a_{rs} B_{rk} C_{is},$$

or, if we remember that

$$A^{n-2} a_{rs} = \frac{d}{d A_{rs}} \{ \mid A_{ik} \mid \},$$

$$A^{n-2} g_{ik} = (-1)^n \begin{vmatrix} A_{11} & \dots & A_{1n}, & B_{1k} \\ \dots & \dots & \dots & \\ A_{n1} & \dots & A_{nn}, & B_{nk} \\ C_{i1} & \dots & C_{in}, & 0 \end{vmatrix} \quad \dots\dots\dots\dots\dots (8).$$

The equation (8) is of course only the form which the equation (4) of § 7 assumes when $m = n-1$.

But now, if P_{ik} denote the minor of p_{ik} in the determinant p, it is clear, from the last equation of § 6, that

$$P_{ik} = A^{n-2} g_{ik}$$
$$= (-1)^n \begin{vmatrix} A_{11} & \dots & A_{1n}, & B_{1k} \\ \dots & \dots & \dots & \\ A_{n1} & \dots & A_{nn}, & B_{nk} \\ C_{i1} & \dots & C_{in}, & 0 \end{vmatrix} .$$

The ratio of complementary minors of the systems p_{ik} and P_{ik} is a power of p; *i.e.*, a power of A multiplied by a power of BC.

13. The last paragraph suggests another form of stating the theorem of § 9: namely, let us take the determinant

$$F_{11} = \begin{vmatrix} A_{11} & \dots & A_{1n}, & B_{11} & \dots & B_{1m} \\ \dots & \dots & \dots & \dots & \dots & \\ A_{n1} & \dots & A_{nn}, & B_{n1} & \dots & B_{nm} \\ C_{11} & \dots & C_{1n}, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ C_{m1} & \dots & C_{mn}, & 0 & \dots & 0 \end{vmatrix} ,$$

and transform it as we did the determinant (6) in § 9. We see that

$$F_{11} = (-1)^n A^{n-m-1} (BC)^{m-1} \begin{vmatrix} a_{11} & \dots & a_{1n}, & b_{1\,m+1} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots & \dots & \\ a_{n1} & \dots & a_{nn}, & b_{n\,m+1}, & \dots & b_{nn} \\ c_{m+1\,1} & \dots & c_{m+1\,n}, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ c_{m1} & \dots & c_{nn}, & 0 & \dots & 0 \end{vmatrix}$$
$$= (-1)^n A^{n-m-1} (BC)^{n-1} g_{11}.$$

And generally, if F_{ik} be formed from the systems A_{ik}, B_{ik}, C_{ik} in the same way in which f_{ik} is formed from the systems a_{ik}, b_{ik}, c_{ik}, we have

$$F_{ik} = (-1)^n A^{n-m-1} (BC)^{m-1} g_{ik}.$$

Then the theorem of § 9 may be stated in the form: the ratio of complementary minors from the systems f_{ik} and F_{ik} is equal to a power of A multiplied by a power of BC.

January 11th, 1883.

Prof. HENRICI, F.R.S., President, in the Chair.

Mr. H. T. Gerrans, B.A., Fellow of Worcester College, Oxford, and Mr. W. L. Mollison, M.A., Fellow of Clare College, Cambridge, were elected Members of the Society.

Dr. Hirst, F.R.S., spoke on "The Resolution of Congruences into Systems of Quadric Reguli"; Mr. J. W. L. Glaisher, F.R.S., discussed "a Differential Equation allied to Riccati's"; and Mr. Tucker communicated a paper by Prof. Cayley, F.R.S., "On the Automorphic Transformation of a Binary Cubic Function."

The following presents were received:—

"Educational Times," January, 1883.

"Johns Hopkins University Circular," Vol. ii., No. 19, November, 1882.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 2° Série, Tome v, 1er cahier; 8vo, Paris, 1882.

"Acta Mathematica," No. 1; 4to, Stockholm, 1882.

"Jahrbuch über die Fortschritte der Mathematik," 12th Band, Jahrgang 1880, Heft 3; Berlin, 1883.

"Washington Astronomical Observations for 1878,—Appendix I., Monograph of the central parts of the Nebula of Orion," by E. S. Holden; Washington, 1882.

On the Automorphic Transformation of the Binary Cubic Function. By Prof. CAYLEY.

[*Read Jan. 11th*, 1883.]

I consider the cubic equation $(a, b, c, d\chi x, 1)^3 = 0$. It is shown (Serret, *Cours d'Algèbre supérieure*, 4th ed., Paris, 1879, t. ii., pp 466–471) how, given one root of the equation, the other two roots can be each of them expressed rationally in terms of this root and of the square root of the discriminant; viz., making the proper changes of notation, and writing

$$A, B, C = ac - b^2,\ ad - bc,\ bd - c^2,\ \lambda = \sqrt{-\tfrac{1}{3}};$$

$$\Omega = B^2 - 4AC,\ = a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd,$$

$$\alpha = \frac{-\lambda\sqrt{\Omega} + B}{2\lambda\sqrt{\Omega}},\ \beta = \frac{+2C}{2\lambda\sqrt{\Omega}},\ \gamma = \frac{-2A}{2\lambda\sqrt{\Omega}},\ \delta = \frac{-\lambda\sqrt{\Omega} - B}{2\lambda\sqrt{\Omega}},$$

(values which give $\alpha + \delta = -1$, $\alpha\delta - \beta\gamma = -1$, and therefore also

$$\alpha^2 + \alpha\delta + \delta^2 + \beta\gamma = 0,$$

which is the condition in order that the function $\phi x, = \frac{\alpha x + \beta}{\gamma x + \delta}$, may

be periodic of the third order, $\phi^3 x = x$), then, u being a root of the equation, say $(a, b, c, d\mathcal{X}u, 1)^3 = 0$, the other two roots are

$$\phi u, = \frac{\alpha u + \beta}{\gamma u + \delta},$$

and
$$\phi^2 u, = \phi^{-1} u, = \frac{(\alpha^2 + \beta\gamma)\, u + \beta\,(\alpha + \delta)}{\gamma\,(\alpha + \delta)\, u + \delta^2 + \beta\gamma}, \quad = \frac{\delta u - \beta}{-\gamma u + \alpha},$$

where observe that, by the change of $\sqrt{\Omega}$ into $-\sqrt{\Omega}$, α, β, γ, δ become δ, $-\beta$, $-\gamma$, α; so that the last-mentioned value $\phi^{-1}u$ is, in fact, the value obtained from ϕu by the mere change of sign of the radical.

It is to be observed that, if we have between two roots u, v of the equation $(a, b, c, d\mathcal{X}x, 1)^3 = 0$, a relation $v = \frac{\alpha u + \beta}{\gamma u + \delta}$, where α, β, γ, δ have given values, this implies in the first place a relation between a, b, c, d (and the given values of α, β, γ, δ), and it implies moreover that u, and consequently also v, are not any roots whatever, but two determinate roots of the equation; viz., u, v will be each of them expressible rationally in terms of a, b, c, d and α, β, γ, δ. And if, in order that (a, b, c, d) may remain arbitrary, we consider α, β, γ, δ as given quantities satisfying the relation which exists between these quantities and (a, b, c, d), then in general we still have u, v determinate roots of the cubic equation. But in the foregoing solution u is any root whatever of the cubic equation.

To examine how this is, starting from the equations

$$(a, b, c, d\mathcal{X}u, 1)^3 = 0, \quad (a, b, c, d\mathcal{X}v, 1)^3 = 0, \quad v = \frac{\alpha u + \beta}{\gamma v + \delta},$$

we have $$a\,(u^3 - v^3) + 3b\,(u^2 - v^2) + 3c\,(u - v) = 0,$$

and therefore $$a\,(u^2 + uv + v^2) + 3b\,(u + v) + 3c = 0,$$

that is $$av^2 + (au + 3b)\, v + au^2 + 3bu + 3c = 0;$$

or, writing herein for v its value,

$$\alpha\,(\alpha u + \beta)^2 + (\alpha u + \beta)(\gamma u + \delta)(au + 3b) + (\gamma u + \delta)^2\,(au^2 + 3bu + 3) = 0;$$

that is,
$$\begin{aligned} \alpha\,(\alpha u + \beta)^2 + \quad & \alpha\gamma\ (\alpha u^3 + 3bu^2) + \ \gamma^2\,(au^4 + 3bu^3 + 3cu^2) = 0; \\ & + (\alpha\delta + \beta\gamma)(au^2 + 3bu) \ + 2\gamma\delta\,(au^3 + 3bu^2 + 3cu) \\ & + \quad \beta\delta\ (au + 3b) \ + \ \delta^2\,(au^3 + 3bu + c) \end{aligned}$$

or, reducing by the equation $au^3 + 3bu^2 + 3cu + d = 0$, this is

$$\begin{aligned} \alpha\,(\alpha u + \beta)^2 + \quad & \alpha\gamma\,(-3cu - d) + \ \gamma^2\,(-du) \qquad = 0, \\ & + (\alpha\delta + \beta\gamma)(au^2 + 3bu) + 2\gamma\delta\,(-d) \\ & + \quad \beta\delta\ (au + 3b) \ + \ \delta^2\,(au^2 + 3bu + 3c) \end{aligned}$$

and, collecting the terms, this is

$$\begin{aligned} & u^2a\,(\alpha^3+\alpha\delta+\delta^2+\beta\gamma) \\ +\,& u\,\{a\,(2\alpha\beta+\beta\delta)+3b\,(\alpha\delta+\delta^2+\beta\gamma)-3c\alpha\gamma-d\gamma^2\} \\ +\,& \quad a\beta^2+3b\beta\delta+3c\delta^2+d\,(-\alpha\gamma-2\gamma\delta)=0. \end{aligned}$$

We can, from this equation, and the equation $au^3+3bu^2+3cu+d=0$, eliminate u, thus obtaining a relation between a, b, c, d, α, β, γ, δ; and, this relation being satisfied, the two equations then determine u rationally in terms of these quantities.

We may without loss of generality assume $\alpha\delta-\beta\gamma=1$; and, this being so, if we then further assume $\alpha+\delta=-1$, then we have

$$\alpha^2+\alpha\delta+\delta^2+\beta\gamma=0$$

(which is, as above appearing, the condition for $\varphi^3x=0$). The foregoing equation in u thus becomes

$$\begin{aligned} & u\,\{a\beta\,(\alpha-1)-3b\alpha^2-3c\alpha\gamma-d\gamma^2\} \\ +\,& \{a\beta^2+3b\beta\delta+3c\delta^2-d\gamma\,(\delta-1)\}=0\,; \end{aligned}$$

a linear equation giving (in a simplified form) the like results to those given by the quadric equation; viz., substituting in the cubic equation the value of u given by the linear equation, we have a relation between a, b, c, d, α, β, γ, δ; and, this relation being satisfied, u has the determinate value given by the linear equation.

The only way in which u can cease to have this determinate value, and so be capable of being any root whatever of the cubic equation, is when the linear equation becomes $0=0$; viz., if

$$\begin{aligned} & a\beta\,(\alpha-1)-3b\alpha^2-3c\alpha\gamma-d\gamma^2 = 0, \\ & a\beta^2+3b\beta\delta+3c\delta^2-d\gamma\,(\delta-1)=0, \end{aligned}$$

equations which are, in fact, satisfied by the foregoing values of α, β, γ, δ, as may be verified without difficulty.

It is to be remarked that, if instead of the root u and the equation $v=\dfrac{\alpha u+\beta}{\gamma u+\delta}$, we consider the root v and the equation $u=\dfrac{\delta u-\beta}{-\gamma u+\alpha}$; then, instead of α, β, γ, δ, we have δ, $-\beta$, $-\gamma$, α, and the corresponding equations are

$$\begin{aligned} & a\beta\,(\delta-1)+3b\delta^2+3c\gamma\delta+d\gamma^2 = 0, \\ & a\beta^2+3ba\beta+3ca^2+d\gamma\,(a-1)=0, \end{aligned}$$

equations which are also satisfied by the foregoing values of α, β, γ, δ. And the four equations, together with $\alpha\delta-\beta\gamma=1$ and $\alpha+\delta=1$, are more than sufficient to determine the foregoing values of α, β, γ, δ.

But we further verify without difficulty that the foregoing values of α, β, γ, δ give identically

$$(a,\,b,\,c,\,d\,(\alpha x+\beta y,\ \gamma x+\delta y)^3=+(a,\,b,\,c,\,d\,(x,\,y)^3;$$

or the formulæ lead to an automorphic transformation of the binary cubic $(a, b, c, d\mathcal{X}x, y)^3$. And conversely, starting from the notion of the automorphic transformation of the binary cubic, we ought to be able to obtain the foregoing formulæ.

For greater convenience, I write the equation of transformation in the form $(a, b, c, d\mathcal{X}\alpha x+\beta y, \gamma x+\delta y)^3 = -\theta\,(a, b, c, d\mathcal{X}x, y)^3$; the equations to be satisfied by $\alpha, \beta, \gamma, \delta, \theta$ then are

$$\begin{aligned}
&a\alpha^3 &&+b\,.\,3\alpha^2\gamma &&+c\,.\,3\alpha\gamma^2 &&+d\gamma^3 &&= -a\theta,\\
&a\alpha^2\beta &&+b\,(\alpha^2\delta+2\alpha\beta\gamma) &&+c\,(2\alpha\gamma\delta+\beta\gamma^2) &&+d\gamma^2\delta &&= -b\theta,\\
&a\alpha\beta^2 &&+b\,(2\alpha\beta\delta+\beta^2\gamma) &&+c\,(\alpha\delta^2+2\beta\gamma\delta) &&+d\gamma\delta^2 &&= -c\theta,\\
&a\beta^3 &&+b\,.\,3\beta^2\delta &&+c\,.\,3\beta\delta^2 &&+d\delta^3 &&= -d\theta.
\end{aligned}$$

Writing $\alpha\delta-\beta\gamma = \nabla$, and as before Ω for the discriminant, the theory of invariants gives $\Omega\nabla^6 = \Omega\theta^4$. We are considering the case of the general cubic function $(a, b, c, d\mathcal{X}x, y)^3$, for which Ω is not $= 0$; and we have therefore $\nabla^6-\theta^4 = 0$, or, what is the same thing, we may write $\nabla = q^2$, $\theta = q^3$, where q is arbitrary.

It is to be shown that $\alpha+\delta$ is $= q$ or $-2q$, the latter value giving the trivial solution $\alpha x+\beta y,\ \gamma x+\delta y = (x, y)$. The proper solution thus corresponds to $\nabla = q^2$, $\alpha+\delta = q$, that is,

$$(\alpha+\delta)^2-(\alpha\delta-\beta\gamma) = 0, \text{ or } \alpha^2+\delta^2+\alpha\delta+\beta\gamma = 0,$$

the condition for the periodic function $\varphi^3 x-x = 0$.

For this purpose, from the foregoing equations eliminating a, b, c, d, we have

$$\begin{vmatrix}
a^3+\theta, & 3a^2\gamma, & 3a\gamma^2, & \gamma^3\\
a^2\beta, & a^2\delta+2a\beta\gamma+\theta, & 2a\gamma\delta+\beta\gamma^2, & \gamma^2\delta\\
a\beta^2, & 2a\beta\delta+\beta^2\gamma, & a\delta^2+2\beta\gamma\delta+\theta, & \gamma\delta^2\\
\beta^3, & 3\beta^2\delta, & 3\beta\delta^2, & \delta^3+\theta
\end{vmatrix} = 0;$$

an equation which may be written

$$\square+\theta\,(123+234+341+412)+\theta^2\,(12+23+34+41+13+42)+\theta^3\,(1+2+3+4)+\theta^4 = 0,$$

where 123, &c. are the first diagonal minors, 12, &c. the second diagonal minors, 1, &c. the third diagonal minors, or diagonal terms of the foregoing determinant, writing therein $\theta = 0$. We find without difficulty

$$\begin{aligned}
1, 2, 3, 4 &= a^3,\ a^2\delta+2a\beta\gamma,\ a\delta^2+2\beta\gamma\delta,\ \delta^3,\\
12, 13, 14, 23, 24, 34 &= \{a^4,\ a^2\,(a\delta+3\beta\gamma),\ (a^2\delta^2+a\delta\beta\gamma+\beta^2\gamma^2),\\
&\qquad (a^2\delta^2+a\delta\beta\gamma+\beta^2\gamma^2),\ \delta^2(a\delta+3\beta\gamma),\ \delta^4\}\,\nabla,\\
123, 124, 134, 234 &= \{a^3,\ a(a\delta+2\beta\gamma),\ \delta(a\delta+2\beta\gamma),\ \delta^3\}\,\nabla^3,\\
\square &= \nabla^6,
\end{aligned}$$

and the equation thus is

$$\begin{aligned}&\theta^4\\ &+\theta^3[\alpha^3+\alpha^2\delta+\alpha\delta^2+\delta^3+2\alpha\beta\gamma+2\beta\gamma\delta]\\ &+\theta^2\nabla[\alpha^4+\alpha^3\delta+\alpha\delta^3+\delta^4+3\alpha^2\beta\gamma+3\beta\gamma\delta^2+2\alpha^2\delta^2+2\alpha\delta\beta\gamma+2\beta^2\gamma^2]\\ &+\theta\ \nabla^3[\alpha^3+\delta^3+(\alpha+\delta)(\alpha\delta+2\beta\gamma)]\\ &+\ \nabla^6=0.\end{aligned}$$

Putting herein $\alpha+\delta=m$, $\alpha\delta=n$, $\beta\gamma=n-\nabla$, it is found that n disappears altogether from the equation; viz., the resulting form is

$$\theta^4+\theta^3 m\,(m^2-2\nabla)+\theta^2\nabla(m^4-3\nabla m^2+2\nabla^2)$$
$$+\theta\nabla^3 m\,(m^2-2\nabla)+\nabla^6=0,$$

or, what is the same thing,

$$m^4.\,\theta^2\nabla+m^3(\theta^3+\theta\nabla^3)-3\theta^2\nabla^2m^2$$
$$-2\,(\theta\nabla^4+\theta^3\nabla)\,m+\theta^4+2\theta^2\nabla^3+\nabla^6=0.$$

Or, putting for θ, ∇, their values q^3, q^2, the equation divides by q^8, and omitting this factor it becomes

$$m^4+2m^3q-3m^2q^2-4mq^3+4q^4=0;$$

viz., this is $\{(m-q)(m+2q)\}^2=0$,

or we have $m=q$ or $-2q$; that is, $\alpha+\delta=q$, or $\alpha+\delta=-2q$.

Writing, as before, A, B, $C=ac-b^2$, $ad-bc$, $bd-c^2$, we deduce from the foregoing equations

$$\begin{aligned}\theta^2A&=\nabla^2[A\,\alpha^2\ \ +B.\alpha\gamma\ \ \ \ +C\gamma^2],\\ \theta^2B&=\nabla^2[A.2\alpha\beta+B\,(\alpha\delta+\beta\gamma)+C.2\gamma\delta],\\ \theta^2C&=\nabla^2[A\,\beta^2\ \ +B.\beta\delta\ \ \ \ +C\delta^2],\end{aligned}$$

which are, in fact, the equations for the automorphic transformation of the Hessian $(A, B, C)(x, y)^2$. And, writing herein θ, $\nabla=q^3$, q^2, the equations become

$$\begin{aligned}A(\alpha^2-q^2)+B\alpha\gamma\ \ \ \ \ \ \ \ &+C.\gamma^2\ \ \ \ \ =0,\\ A\,2\alpha\beta\ \ \ +B(\alpha\delta+\beta\gamma-q^2)+C.2\gamma\delta\ \ \ &=0,\\ A\beta^2\ \ \ \ \ \ +B\beta\delta\ \ \ \ \ \ \ \ &+C(\delta^2-q^2)=0.\end{aligned}$$

From the first and second of these we have

$$A:B:C=\gamma^2(\nabla+q^2):-2\gamma\alpha\nabla+2\gamma\delta q^2:\alpha^2\nabla-(\alpha^2+\alpha\delta+\beta\gamma)\,q^2+q^4;$$

or, writing herein for ∇, q^4 the values q^2, $q^2(\alpha\delta-\beta\gamma)$, the three expressions divide by $2\gamma q^2$, and we have

$$A:B:C=\gamma:\delta-\alpha:-\beta.$$

Combining these values in the first place with the equation $\alpha+\delta=-2q$, we may write $\alpha,\ \beta,\ \gamma,\ \delta=-q-pB,\ -2pC,\ 2pA,\ -q+pB$,

where p is to be determined. Substituting in the last of the three equations, we have

$$A \,.\, 4p^3C^2 - 2pBC\,(-q+pB) + C\,(-2pqB+p^2B^2) = 0,$$

that is, $$p^2 C\,(4AC-B^2), = -p^2 \,.\, C\Omega, = 0,$$

and the form $(a, b, c, d\mathcal{X}x, y)^3$ being arbitrary, neither C nor Ω is $= 0$; whence $p = 0$, and the values are $\alpha, \beta, \gamma, \delta = -q, 0, 0, -q$, that is,

$$(a, b, c, d\mathcal{X} - qx, -qy)^3 = -q^3 \,.\, (a, b, c, d\mathcal{X}x, y)^3,$$

a trivial result.

But, combining the same values with $\alpha+\delta = q$, we have

$$\alpha, \beta, \gamma, \delta = \tfrac{1}{2}q - pB, \; -2pC, \; 2pA, \; \tfrac{1}{2}q + pB;$$

and then substituting in the third equation

$$A \,.\, 4p^2C^2 - 2pBC\,(\tfrac{1}{2}q+pB) + C\,(-\tfrac{3}{4}q^2 + pqB + p^2B^2) = 0,$$

that is, $$C\,\{(4AC-B^2)\,p^2 - \tfrac{3}{4}q^2\} = 0,$$

or, omitting the factor C, and introducing the foregoing notation $\lambda^2 = -\frac{1}{3}$, this is $4\Omega\lambda^2 p^2 - q^2 = 0$, or say $p = \dfrac{1}{2\lambda\sqrt{\Omega}}\, q$. For the unimodular substitution $\alpha\delta - \beta\gamma = 1$, we must have $q^2 = 1$; but, the transformation being

$$(a, b, c, d\mathcal{X}\alpha x+\beta y, \; \gamma x+\delta y)^3 = -q^3 \,.\, (a, b, c, d\mathcal{X}x, y)^3,$$

to make this strictly automorphic, we must have $q^3 = -1$, and the two conditions are satisfied by $q = -1$; we then have $p = -\dfrac{1}{2\lambda\sqrt{\Omega}}$; and the coefficients are

$$\alpha, \; \beta, \; \gamma, \; \delta, = \frac{-\lambda\sqrt{\Omega}+B}{2\lambda\sqrt{\Omega}}, \; \frac{2C}{2\lambda\sqrt{\Omega}}, \; \frac{-2A}{2\lambda\sqrt{\Omega}}, \; \frac{-\lambda\sqrt{\Omega}-B}{2\lambda\sqrt{\Omega}};$$

which are the values given at the beginning of the paper, and which belong to the automorphic transformation

$$(a, b, c, d\mathcal{X}\alpha x+\beta y, \; \gamma x+\delta y)^3 = (a, b, c, d\mathcal{X}x, y)^3.$$

The *à priori* reason for the periodicity-equation $\phi^3 x = x$, is best seen by expressing the cubic function as a product of factors

$$M\,(x - \mathrm{a}y)(x - \mathrm{b}y)(x - \mathrm{c}y).$$

The substitution must, it is clear, cyclically interchange these factors, and therefore, when performed three times in succession on any one of these factors, or consequently upon an arbitrary linear factor $x - \mathrm{f}y$, must leave the factor unaltered, and it must thus be a periodic substitution $\phi^3 x = x$. But it was interesting to see how the condition for this $\alpha^2 + \delta^2 + \alpha\delta + \beta\gamma = 0$, comes out from the consideration of the equation $(a, b, c, d\mathcal{X}\alpha x + \beta y, \; \gamma x + \delta y)^3 = (a, b, c, d\mathcal{X}x, y)^3$.

February 8th, 1883.

Prof. HENRICI, F.R.S., President, in the Chair.

Captain P. A. Macmahon, R.A., was admitted into the Society. The Auditor (Mr. J. Hammond) made his report. On the motion of the Chairman, a vote of thanks was unanimously accorded to the Auditor for his services.

The following communications were made :—

"On the Sylvester-Kempe Quadruplane," Mr. H. Hart.

"On Curves obtained by an extension of Maclaurin's method of constructing Conics," Mr. S. Roberts, F.R.S.

"A Generalisation of the Nine-point Properties of a Triangle," Captain P. A. Macmahon, R.A.

"On the Use of certain Differential Operators in the Theory of Equations," Mr. J. Hammond.

"A method for reducing the Differential Expression

$$dt \,/\, \surd\{t-\alpha\,.\,t-\beta\,.\,t-\gamma\,.\,t-\delta\}$$

to the Standard Form," Mr. J. Griffiths.

The following presents were received :—

"Educational Times," February, 1883.

"Proceedings of the Royal Society," Vol. xxxiv., No. 222.

"Methods of Election," by Prof. E. J. Nanson; a Paper read before the Royal Society of Victoria; Octavo pamphlet.

"Johns Hopkins University Circular," Vol. ii., No. 20.

"Scientific Proceedings of the Ohio Mechanics' Institute," Vol. i., No. 4; Octavo.

"Società Reale di Napoli,—Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche," Vols. 1 to 20; Naples, 4to.

"Bulletin des Sciences Mathématiques et Astronomiques," August, September, October, and November, 1882.

"Beiblätter zu den Annalen der Physik und Chemie," 1882, Band vi., No. 12; 1883, Band vii., No. 1.

"Atti della R. Accademia dei Lincei—Transunti," Vol. vii., Fasc. 1 and 2.

"Crelle," Band xciii., Heft 4.

"Journal de l'École Polytechnique," &c., 51st cahier; Paris, 1882.

"Bulletin Astronomique et Météorologique de l Observatoire Impérial de Rio Janeiro," October, 1882, No. 10; November, 1882, No. 11.

"Acta Mathematica," No. 1, 4to; Stockholm, 1882

"Programma dei Concorsi ai premj proposti dal R. Istituto Lombardo," &c., 1883; 8vo.

"Bulletin de la Société Mathématique de France," Tome x, No. 7; Paris, 1882.

"Annali di Matematica," Serie ii^a, Tomo xi°, Fasc. 2° e 3°; Milano, 1883.

On Curves obtained by an extension of Maclaurin's method of constructing Conics. By Samuel Roberts.

[*Read Feb. 8th*, 1883.]

In Maclaurin's construction, the points of a conic are marked by the vertex of a variable triangle, whose sides pass respectively through fixed points, and whose remaining vertices move on given straight lines. The generalisation above referred to is briefly noticed in Salmon's *Higher Plane Curves* (1st edition, p. 247), where the degree of the locus is given in several comprehensive cases. In what follows, I treat the subject in rather more detail by determining the degree and class of the loci under slightly more general conditions. There are special cases which seem to be of some interest; but many of these are of a simple character, and have been elsewhere given in a more or less isolated way. I confine myself generally to conditions which do not lead to very complicated singularities. Analogous results could probably be obtained in more intricate cases, if for any special purpose it became desirable. Thus, instead of ordinary double points or multiple points on a curve directrix, we might assume singularities equivalent to a certain number of double points and cusps, and so forth. It will often happen that, under such circumstances, the degree and class will still be given by the general formula.

For shortness I use the symbol C_m^n to denote a curve of the m^{th} degree and n^{th} class.

I.—If the sides BC, CA, AB of a triangle ABC pass respectively through the fixed points P, Q, R, while the vertices B, A move respectively on the curves $C_{m_1}^{n_1}$, $C_{m_2}^{n_2}$, and if P is multiple on the curve $C_{m_1}^{n_1}$ to the order p, Q is multiple on the curve $C_{m_2}^{n_2}$ to the order q, and R is multiple on the curve $C_{m_1}^{n_1}$ to the order r, and on the curve $C_{m_2}^{n_2}$ to the order r'; then the degree of the locus of the vertex C is

$$(m_1-p)\,(m_2-r')+(m_2-q)\,(m_1-r),$$

and the class is

$$2\,(m_1-p-r)\,(m_2-q-r')+n_1\,(m_2-r')+n_2\,(m_1-r)-2pq.$$

The points P, Q, R are supposed to be simply multiple on the directrices.

The circumstance to be specially noticed here and in subsequent cases, is that the degree of the locus is not affected by point singularities on the curve directrices, except when they exist in special situations, as for example at the points P, Q, R, or on the line joining

P, Q. This appears by the actual process of the generation, for point singularities on the directrices in general produce point singularities on the locus.

Let us take then, for $C^{n_1}_{m_1}$, m_1 straight lines, of which p pass through P and r through R; for $C^{n_2}_{m_2}$, m_2 straight lines, of which q pass through Q and r' through R.

If $m_1 = m_2 = 1$, $n_1 = n_2 = 0$, we have Maclaurin's construction for conics. If, however, P or R is on $C^{n_1}_{m_1}$, Q or R on $C^{n_2}_{m_2}$, the locus is a straight line through the point. If both P and R or Q and R are on the directrix of B in the first instance, or of A in the second, the locus is still a straight line. If P is on the directrix of B, and Q on that of A, or if the directrices meet in R, the degree is zero. The fact of P being on the directrix of A, or Q on that of B, does not affect the degree.

There results then, for the general degree,

$$\begin{aligned} &2\,(m_1-p-r)\,(m_2-q-r')+(m_1-p)\,q+(m_2-q)\,p \\ &\qquad +(m_1-p-r)\,r'+(m_2-q-r')\,r \\ &=(m_1-p)\,(m_2-r')+(m_2-q)\,(m_1-r). \end{aligned}$$

We may now consider the general curves $C^{n_1}_{m_1}$, $C^{n_2}_{m_2}$. If any transversal is drawn through P, there are on it only $(m_1-p)(m_2-r')$ points of the locus distinct from P. Hence P is a multiple point of the order $(m_2-q)\,(m_1-r)$, and by symmetry Q is a multiple point of the order $(m_1-p)\,(m_2-r')$. Except when further special conditions are imposed, the tangents at these multiple points will be distinct; that is, the points are ordinary multiple points.

Consider now the tangents which can be drawn to the locus from and at the point P, counting those at P twice.

From P we can draw n_1-2p tangents to $C^{n_1}_{m_1}$ (unless a branch through P is rectilinear, which case is exceptional), and these are multiple tangents to the locus, having in general m_2-r' distinct points of contact. If we draw tangents to $C^{n_2}_{m_2}$ from R, they also are in general n_2-2r' in number (the case of a rectilinear branch through R being exceptional) and determine m_1-r tangents to the locus from P.

This includes all the ways in which tangents to the locus from P arise.

The necessity for excluding the case of rectilinear branches follows from the circumstance that n_1-2p does not express the number of tangents which can be drawn from a point ($p=1$) on a straight line, or from a multiple point of the order p forming the intersection of lines, one or more of which are straight. Sometimes the general expression for the class will in such a case become negative, thus indicating the discontinuity, but it may happen otherwise.

Taking the number of tangents at and from P to the locus, we get the class

$$2\,(m_2-q)\,(m_1-r)+(n_1-2p)\,(m_2-r')+(n_2-2r')\,(m_1-r)$$
$$=2\,(m_1-p-r)\,(m_2-q-r')+n_1\,(m_2-r')+n_2\,(m_1-r)-2pq.$$

Certain results of the generation can be immediately discerned. The intersections of the curve directrices are points on the locus. The straight line joining Q, R meets $C^{n_1}_{m_1}$ in m_1-r points distinct from P, and these are multiple points on the locus of the order m_2-q-r'. Similarly the straight line joining P, R meets $C^{n_2}_{m_2}$ in m_2-r' points distinct from Q, and these are multiple points on the locus of the order m_1-p-r.

The degree of the locus suffers a reduction, if intersections of the directrices lie on the straight line joining P, Q; for the sum of the orders of the multiple points at P and Q is the general degree.

The degree may also suffer a reduction in consequence of transversals through P being common tangents to the directrices. These create additional double points. If there are t such common tangents, the class is reduced by $2t$.

A double point or cusp on $C^{n_1}_{m_1}$ occasions m_2-r' double points or cusps on the locus. *Mutatis mutandis*, the same holds for $C^{n_2}_{m_2}$.

The following particular cases occur :—

If $p=q=r=r'=1$, and $m_1=m_2=2$,
or if $p=r=m_2=1$, $q=r'=0$, $m_1=2$,
or if $q=r'=m_1=1$, $p=r=0$, $m_2=2$,

the locus is a conic.

If $p=q=1$, $r=r'=0$, $t=2$, $m_1=m_2=2$,

the locus is of the fourth degree and fourth class, breaking up into two conics (Salmon's *Conics*, 5th edition, Ex. 7, p. 288).

If $p=r=r'=1$, $q=0$, $m_1=m_2=2$, or if $p=q=r'=1$, $r=0$,
$m_1=m_2=2$, or if $q=r=r'=1$, $p=0$, $m_1=m_2=2$,

the locus is a unicursal cubic.

If $p=q=r=r'=0$, $m_1=m_2=2$, $t=0$,

the locus is of the eighth degree and sixteenth class. But, if $t=2$, the locus is of the twelfth class, breaking up into two unicursal quartics. The points P, Q are double on both; the other two double points correspond to the common tangents to the directrices through R.

If $p=q=1$, $m_1=m_2=2$, $r=r'=0$, and one intersection of the conic directrix lies on the straight line joining P, Q, the locus is of the third degree and sixth class.

The three intersections of the conics not on PQ are on the locus—call them 1, 2, 3. Then P is the coresidual of 1, 2, 3, Q, and Q is the coresidual of 1, 2, 3, P.

II. If in the construction of I. the points P, Q, R are collinear, the degree of the locus of C becomes

$$(m_1-p)(m_2-r')+p(m_2-q-r')+r'(m_1-r)=m_1m_2-pq-rr',$$

and the class is

$$2p(m_2-q-r')+2r'(m_1-r)+(n_1-2p)(m_2-r')+(n_2-2r')(m_1-r)$$
$$=m_2n_1+m_1n_2-rn_2-r'n_1-2pq.$$

When we take two straight lines for directrices, the locus is a straight line, unless P is on the directrix of B, and Q on that of A. If this is the case, the degree of the locus is zero.

Taking then, for $C^{n_1}_{m_1}$, m_1 straight lines of which p pass through P, and for $C^{n_2}_{m_2}$, m_2 straight lines of which q pass through Q, we get for the degree

$$(m_1-p)(m_2-q)+(m_1-p)q+(m_2-q)p,$$

which is the general degree for the case $r=r'=0$.

Now, suppose, instead of straight lines, we take for $C^{n_1}_{m_1}$ M conics, p of which pass through P and r through R; and for $C^{n_2}_{m_2}$ M' conics, of which q pass through Q and r' through R.

For two conic directrices, both passing through R, the locus is a cubic, and so also if $p=q=1$.

The degree for C^{2M}_{2M} and $C^{2M'}_{2M'}$ as directrices is then

$$4\{(M-p-r)(M'-q-r')+(M-p)q+(M'-p)p$$
$$+(M-p-r)r'+(M'-q-r')r\}+3(pq+rr')$$
$$=4MM'-pq-rr'.$$

To meet the case of m_1 or m_2 odd, or both, we may add a straight line to C^{2M}_{2M} or $C^{2M'}_{2M'}$, or both; the addition is $2M$ or $2M'$ or $2(M+M'+1)$ in the respective cases. Hence the general expression is

$$m_1m_2-pq-rr'.$$

As to the class, the number of points on the locus lying on a transversal through P and distinct from P is $(m_1-p)(m_2-r')$. Hence there is a multiple point of the order $p(m_2-q-r')+r'(m-r)$ at P. In general, this will be an ordinary multiple point. Taking the tangents to the locus through P, and counting those at P twice, we have for the class

$$2p(m_2-q-r')+2r'(m-r)+(n_1-2p)(m_2-r')+(n_2-2r')(m-r)$$
$$=m_1n_2+m_2n_1-r'n_1-rn_2-2pq.$$

For a particular case, we have $p=q=1$, $r=r'=0$, $m_1=m_2=2$; the locus is a cubic of the sixth class.

In fact, a general cubic can thus be drawn. For the tangents to the conics at P, Q are tangents to the locus at the same points, and their intersection is the common tangential of P, Q. If then we have given two tangents OP, OQ of a cubic through O, P, Q (P, Q being the points of contact), we can draw a conic having OP for a tangent at P, and passing through three arbitrary points. We can also draw another conic having OQ for a tangent at Q, and passing through the same three points. If we draw two straight lines through P and Q respectively, meeting in a fourth arbitrary point, we determine R. Thus we can describe by the present method a cubic through four arbitrary points, and having at P, Q the tangents OP, OQ, whose common tangential is O. The points O, P, Q are also optional.

The fourth intersection of the conics is common to the cubics of the system passing through the three arbitrary points first taken.

If a transversal through P touches the directrices in points distinct from R, there arises a double point on the locus. The class reduction is $2t$ for t such transversals.

Thus, if $m_1 = m_2 = 2$, $p = q = r = r' = 0$, $t = 2$, the locus breaks up into two conics. If we take for directrices two circles and a centre of similitude R, with two points P, Q collinear with R, one part of the locus is a circle, and the other a conic through the finite intersections of the circle directrices. Otherwise, the two factors of the locus are analogously related.

The result is equivalent (as to the circle factor of the locus) to an elementary property of homologous points on three circles.

The centres of similitude lie collinearly in threes. If we take three homologous points, one on each circle, and join them in pairs by straight lines, these will respectively pass through one of three collinear centres of similitude. Thus, if p, q, r are the homologous points, and P, Q, R the centres of similitude through which the straight lines qr, rp, rq respectively pass, pqr forms a triangle conditioned in the manner supposed in the general construction. Hence, if q, r move on their circles, p describes the third circle. But if we take anti-homologous points for q, r, the locus of p is a conic analogously related to the figure.

I have shown elsewhere* that, if $m_2 = 1$, $p = q = r = r' = 0$, the degree of the locus is m_1 and the class n_1; and that the locus is really a homographic transformation of $C^{n_1}_{m_1}$.

III. We will now suppose that the directrices $C^{n_1}_{m_1}$, $C^{n_2}_{m_2}$, of I., coincide with C^{n}_{m}. The degree of the locus is then

$$(m-p)(m-r-1)+(m-q)(m-r-1) = (2m-p-q)(m-r-1),$$

* Reprint of Mathematics from the *Educational Times*, Vol. xxix., p. 96.

and the class is

$$2(m-q)(m-r-1)+(n-2p)(m-r-1)+(n-2r)(m-r-2).$$

The curve C_m^n is assumed to have a multiple point of the order p at P, one of the order q at Q, and one of the order r at R.

Take for C_m^n M conics, of which p pass through P, q through Q, and r through R; these conics may be taken as distinct. Then, remembering that, in the case of a pair of conic directrices, either may be a directrix of B and also of A, we find for the degree

$$\begin{aligned}
&(8+8)\,\frac{(M-p-q-r)(M-p-q-r-1)}{2}\\
&+(8+6)(M-p-q-r)(p+q)+4(M-p-q-r)\\
&+(8+4)\{(M-p-q-r)r+pq\}\\
&+(6+6)\left\{\frac{p(p-1)}{2}+\frac{q(q-1)}{2}\right\}+(4+4)\left(\frac{r\,.\,r-1}{2}\right)\\
&+(5+6)\{p+q\}r+3(p+q+r)\\
&\quad=8(M-p-q-r)(M-p-q-r-1)\\
&\qquad+(M-p-q-r)(14p+14q+12r+4)\\
&\qquad+4r(r-1)+6\{p(p-1)+q(q-1)\}+11(p+q)r\\
&\qquad\quad+3(p+q+r)+12pq\\
&\quad=8M^2-4M-2M(p+q+2r)+p+q+pr+qr\\
&\quad=2m(m-1)-m(p+q+2r)+p+q+pr+qr,\ \text{if } 2M=m.
\end{aligned}$$

To meet the case of m odd, we may add a straight line to C_m^n; the addition to the degree is

$$8(M-p-q-r)+(4+3)p+(4+3)q+(3+3)r=8M-p-q-2r,$$

giving the same result.

The points of the locus distinct from P on a transversal through that point, are in number $(m-p)(m-r-1)$. There is, therefore, a multiple point of the order $(m-q)(m-r-1)$ at P; and similarly one of the order $(m-p)(m-r-1)$ at Q.

Counting tangents from and at P as in previous cases, we find for the class

$$2(m-q)(m-r-1)+(n-2p)(m-r-1)+(n-2r)(m-r-2).$$

If $m=2$, $p=q=1$, $r=0$, the locus is a conic. This is an instructive example, for the locus remains the same for any conic directrix passing through P, Q, and any two points on the original directrix collinear with R. It remains the same, therefore, for an infinite number of straight directrices in pairs. In fact, we may take as directrices the two sides of any generating triangle, which pass through P and Q respectively. The locus is that of the points of

contact of tangents from R to the system of conic directrices. This is an instance in which the class formula fails for the straight directrices, $n-2p$ being negative and illusory.*

IV. If, in III., P, Q, R are collinear, the degree is

$$(m-p)(m-r-1)+p(m-q-r-1)+r(m-r-1)$$
$$= m(m-1)-pq-r(r+1),$$

and the class is

$$2p(m-q-r-1)+2r(m-r-1)+(n-2p)(m-r-1)+(n-2r)(m-r-2)$$
$$= n(2m-3)-2(n-1)r-2pq.$$

As in the preceding case, we take for C_m^n M conics. Then the degree is

$$(4+4)\left\{\frac{(M-p-q-r)(M-p-q-r-1)}{2}+(M-p-q-r)(p+q+r)+qr+pr\right\}+$$

$$(4+3)pq+2(M-p-q-r)+3r(r-1)+4p(p-1)+4q(q-1)+2p+2q$$
$$= 2M(2M-1)-pq-r(r+1).$$

We may again meet the case of m odd by adding a straight line. The consequent addition is

$$4(M-p-q-r)+(2+2)(p+q+r)=4M.$$

The general degree then is

$$m(m-1)-pq-r(r+1).$$

On a transversal through P, there are $(m-p)(m-r-1)$ points of the locus distinct from P. There is then at P a multiple point of the order $p(m-q-r-1)+r(m-r-1)$. As in previous cases, we get the class above given.

If $m=n=2$, $p=q=1$, $r=0$, the locus is a straight line, evidently the polar of R with respect to the conic.

If $m=3$, $n=6$, $p=q=r=1$, the locus is also a cubic of the sixth class.

---

* The lines RP, RQ are tangents to the locus at P, Q. Two points on the conic directrix collinear with R determine, therefore, a fifth point of the conic locus. Analytically, the result comes out as follows:—

Let RP, PQ, QR be the sides β, γ, α of the triangle of reference, and $C\gamma^2+2D\beta\gamma+2E\alpha\gamma+2F\alpha\beta=0$ the equation of the conic directrix.

Then, if a tranversal through R (α, β) meets the conic in the points (α', β'), (α'', β''), we have, putting $\alpha-k\beta=0$ for the equation of the transversal,

$$\frac{\beta'\beta''}{\gamma'\gamma''}=\frac{C}{2Fk}=\frac{\beta'\alpha''}{\gamma'\gamma''k};$$

and X, Y, Z being the current coordinates, so that

$$Z\beta'-Y\gamma'=0, \quad Z\alpha''-X\gamma''=0,$$

we have $CZ^2-2FXY=0$ independently of D, E.

V. If the p sides of a polygon pass through fixed points respectively and all the vertices but one move on curves $C^{n_1}_{m_1}$, $C^{n_2}_{m_2} \dots C^{n_{p-1}}_{m_{p-1}}$ respectively, the degree of the locus of the remaining vertex is $2m_1m_2 \dots m_{p-1}$, and the class is

$$2m_1m_2 \dots m_{p-1} + n_1m_2m_3 \dots m_{p-1} + n_2m_1m_3 \dots m_{p-1} + \dots + n_{p-1}m_1 \dots m_{p-2}.$$

It may be shown (Salmon's *Higher Plane Curves*, 1st ed., p. 247) that on a transversal through (say) the fixed point on a side of the describing vertex, there are $m_1m_2 \dots m_{p-1}$ points of the locus distinct from that point, and that the point itself is multiple in the locus to the order $m_1m_2 \dots m_{p-1}$. Or we can separate the polygon into triangles. Thus, if the polygon is a quadrilateral $M_1M_2M_3C$, C being the describing vertex, let P_1, P_2, P_3, P_4 be the fixed points through which pass respectively CM_1, M_1M_2, M_2M_3, M_3C. Produce M_1C and M_2M_3 to meet in K. Then the locus of K is of the degree $2M_1M_2$, with P_1, P_3 for multiple points of the order m_1m_2.

Again, the locus of C is, by the formula of I.,

$$m_3(2m_1m_2 - m_1m_2) + m_3(2m_1m_2 - m_1m_2) = 2m_1m_2m_3.$$

We can use the same process for the general polygon. As to the class, for the quadrilateral, the class of the locus of K is

$$2m_1m_2 + m_1n_2 + m_2n_1,$$

and the class of the locus of C is, by III.,

$$2(2m_1m_2 - m_1m_2 - m_1m_2)m_3 + (2m_1m_2 + n_1m_2 + n_2m_1)m_3 + n_3(2m_1m_2 - m_1m_2)$$
$$= 2m_1m_2m_3 + n_1m_2m_3 + n_2m_1m_3 + n_3m_1m_3.$$

We can now assume the formula for a polygon of $p-1$ sides (taking notice also of the multiple points at two of the fixed points), and thence, as above, show that the same formula holds for a polygon of p sides.

We may, indeed, similarly treat the case in which the fixed points are multiple points on the respective directrices with which they are associated.

It will be sufficient to take the case of a quadrilateral, $M_1M_2M_3C$. Suppose on $C^{n_1}_{m_1}$ that P_1 is a multiple point of the order p_1, and P_2 a multiple point of the order p_2. Let the corresponding orders for P_2, P_3 on $C^{n_2}_{m_2}$, and P_3, P_4 on $C^{n_3}_{m_3}$, be r_2, r_3 and s_3, s_4.

Then the degree of the locus of K is

$$(m_1 - p_1)(m_2 - r_2) + (m_1 - p_2)(m_2 - r_3),$$

and the class is

$$2(m_1 - p_1 - p_2)(m_2 - r_2 - r_3) + n_1(m_2 - r_2) + n_2(m_1 - p_1) - 2p_1r_3, \text{ by I.}$$

The locus of C is then of the degree

$$(m_3 - s_4)(m_1 - p_2)(m_2 - r_3) + (m_3 - s_3)(m_1 - p_1)(m_2 - r_2),$$

and the class is

$$n_3(m_1-p_2)(m_2-r_3)+\{2(m_1-p_1-p_2)(m_2-r_2-r_3)+n_1(m_2-r_2)+n_2(m_1-p_2)-2p_1r_3\}(m_3-s_3)-2s_4(m_1-p_2)(m_2-r_3)$$
$$=2(m_3-s_3-s_4)(m_2-r_3-r_2)(m_1-p_1-p_2)+n_3(m_1-p_2)(m_2-r_3)+n_2(m_1-p_2)(m_3-s_3)+n_1(m_2-r_2)(m_3-s_3)-2m_1s_4r_2-2m_2s_3p_1-2m_3r_3p_1+2s_4p_1r_3+2s_4r_2p_2+2s_4p_1r_2+2p_1r_3s_3.$$

VI. For the fixed points of I., we may substitute curves $C_{M_1}^{N_1}$, $C_{M_2}^{N_2}$, $C_{M_3}^{N_3}$, which are touched respectively by the sides BC, CA, AB of the generating triangle. These curves are supposed not to have special relations to the other directrices.

The degree of the locus is then $2N_1N_2N_3m_1m_2$, and the class is

$$N_1N_2N_3(2m_1m_2+n_1m_2+n_2m_1)+m_1m_2(N_1N_2M_3+N_1N_3M_2+N_2N_3M_1).$$

In fact, if we take for $C_{M_1}^{N_1}$ N_1 points, for $C_{M_2}^{N_2}$ N_2 points, and for $C_{M_3}^{N_3}$ N_3 points, the degree of the locus is $2N_1N_2N_3m_1m_2$ by what precedes, and this result remains unaltered by the double tangents arising from the points N_1 taken two and two, the points N_2 taken two and two, or the points N_3 taken two and two.

A double tangent of this kind does, however, affect the class, since it occasions, if existing on $C_{M_1}^{N_1}$ (now considered as a proper curve), $N_2N_3m_1m_2$ double points on the locus. The case is similar for the other directrices $C_{M_2}^{N_2}$ and $C_{M_3}^{N_3}$. Making allowance for this, and also for the existence of double tangents and inflexions, we must add to the first result $N_1N_2N_3(2m_1m_2+n_1m_2+n_2m_1)$ the following complement

$$m_1m_2(N_1N_2M_3+N_1N_3M_2+N_2N_3M_1).$$

If we generalise III. in a similar fashion, the degree obtained is $2N_1N_2m(m-1)$, and the class is

$$N_1N_2N_3\{n(2m-3)+2m(m-1)\}+m(m-1)(N_1N_2M_3+N_1N_3M_2+N_2N_3M_1).$$

It will, without doubt, be remarked how far the cases which I have dealt with fall short of being exhaustive. Some additional ones could be added if we were content with the determination of the degree of the locus. The determination of the class is accompanied with special difficulties, and the few results I have obtained in this direction are imperfect. The problem, in its ultimate form, requires us to determine the degree and class of the locus when the directrices of the two vertices of the generating triangle, and those which are touched by the sides, coincide in one curve. Intermediate cases lead up to this.

On the use of certain Differential Operators in the Theory of Equations. By J. Hammond.

[*Read Feb. 8th*, 1883.]

1. If $D_1, D_2, D_3, \ldots$ denote operators defined by the equation

$$\phi\left(p_1-\frac{1}{y},\ p_2-\frac{p_1}{y},\ p_3-\frac{p_2}{y},\ \ldots\right)$$
$$=(1-y^{-1}D_1+y^{-2}D_2-y^{-3}D_3+\ldots)\,\phi\,(p_1, p_2, p_3, \ldots)\ldots\ldots\ldots(1),$$

it has been shown in a former paper (*Proceedings*, Vol. xiii., p. 79) that when

$$\phi\,(p_1, p_2, p_3, \ldots)=\Sigma\alpha_1^{\lambda}\alpha_2^{\lambda}\ldots\alpha_l^{\lambda}\beta_1^{\mu}\beta_2^{\mu}\ldots\beta_m^{\mu}\gamma_1^{\nu}\gamma_2^{\nu}\ldots\gamma_n^{\nu}\ldots$$
$$=(\lambda^l.\,\mu^m.\,\nu^n\ldots)\ \text{suppose,}$$

where $\alpha_1, \alpha_2, \ldots \beta_1, \beta_2, \ldots \gamma_1, \gamma_2, \ldots$ are roots of the equation

$$1-x^{-1}p_1+x^{-2}p_2-x^{-3}p_3+\ldots=0\ldots\ldots\ldots\ldots\ldots\ (2),$$

$D_\kappa(\lambda^l.\,\mu^m.\,\nu^n\ldots)=0$, $D_\lambda(\lambda^l.\,\mu^m.\,\nu^n\ldots)=(\lambda^{l-1}.\,\mu^m.\,\nu^n\ldots)$, $D_\lambda(\lambda)=1$.

The operators of the present paper are the symmetric functions of the roots of the equation

$$1-y^{-1}D_1+y^{-2}D_2-y^{-3}D_3+\ldots=0\ \ldots\ldots\ldots\ldots\ (3),$$

considered simply as rational integral functions of $D_1, D_2, D_3, \ldots$, when operating on $(\lambda^l.\,\mu^m.\,\nu^n\ldots)$ and as differential operators on $\phi\,(p_1, p_2, p_3, \ldots)$. It is easy to see that these operators follow the ordinary laws of quantity in their combinations with constants and with each other; *e.g.*, if $\phi\,(p_1, p_2, p_3, \ldots)=(\lambda^l.\,\mu^m.\,\nu^n\ldots)$, we have

$$D_\lambda D_\mu\phi=(\lambda^{l-1}.\,\mu^{m-1}.\,\nu^n\ldots)=D_\mu D_\lambda\phi,$$

and D_λ, D_μ obey the commutative law. In the case of the remaining laws, a formal proof is hardly required.

The meaning of equation (3), or of its roots, does not at all concern us, as they are only used for purposes of concise definition; thus, if

$$1-y^{-1}D_1+y^{-2}D_2-y^{-3}D_3+\ldots=\left(1-\frac{a}{y}\right)\left(1-\frac{b}{y}\right)\left(1-\frac{c}{y}\right)\ldots,$$

and if $\Sigma a^\lambda b^\mu c^\nu\ldots=[\lambda.\,\mu.\,\nu\ldots]$, using $[\ ]$ instead of $(\ )$ to show that the roots referred to are $a, b, \ldots$ instead of $\alpha, \beta, \ldots$, we have

$$D_1=\Sigma a=[1],\quad D_2=\Sigma ab=[1^2],$$
$$D_\kappa=[1^\kappa]\ \text{and}\ D_1^2-2D_2=\Sigma a^2=[2].$$

This notation was suggested by Professor Cayley.

2. A line placed over any expression will be used to denote that all its symbols are to be combined by the ordinary laws of quantity; thus from (1) we obtain directly

$$D_\kappa = \frac{1}{\kappa\,!}\overline{\left(\frac{d}{dp_1}+p_1\frac{d}{dp_2}+p_2\frac{d}{dp_3}+\ldots\right)^\kappa} = \frac{1}{\kappa\,!}\overline{d_1^\kappa} \quad \ldots\ldots\ldots (4),$$

where, for shortness,

$$\left.\begin{aligned} d_1 &= \frac{d}{dp_1}+p_1\frac{d}{dp_2}+p_2\frac{d}{dp_3}+\&c.\\ d_2 &= \frac{d}{dp_2}+p_1\frac{d}{dp_3}+p_2\frac{d}{dp_4}+\&c.\\ d_3 &= \frac{d}{dp_3}+p_1\frac{d}{dp_4}+p_2\frac{d}{dp_5}+\&c.\\ &\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ d_\lambda &= \frac{d}{dp_\lambda}+p_1\frac{d}{dp_{\lambda+1}}+p_2\frac{d}{dp_{\lambda+2}}+\&c. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5).$$

Now $$d_\lambda D_\kappa = \frac{1}{\kappa\,!}d_\lambda\overline{d_1^\kappa} = \frac{1}{\kappa\,!}\overline{d_\lambda d_1^\kappa}+\frac{1}{(\kappa-1)\,!}\overline{d_1^{\kappa-1}d_{\lambda+1}},$$

the last term arising from the differentiation of D_κ considered as an explicit function of $p_1, p_2, p_3, \ldots$

This result may also be written

$$d_\lambda D_\kappa = \overline{d_{\lambda+1}D_{\kappa-1}}+\overline{d_\lambda D_\kappa} \quad \ldots\ldots\ldots\ldots\ldots (6).$$

Reducing by means of (6), we have,

$$\begin{aligned} &d_\lambda - d_{\lambda-1}D_1+d_{\lambda-2}D_2-d_{\lambda-3}D_3+\ldots+(-)^{\lambda-1}d_1D_{\lambda-1}\\ &= d_\lambda-(d_\lambda+\overline{d_{\lambda-1}D_1})+(\overline{d_{\lambda-1}D_1}+\overline{d_{\lambda-2}D_2})-(\overline{d_{\lambda-2}D_2}+\overline{d_{\lambda-3}D_3})+\ldots\\ &\qquad\ldots+(-)^{\lambda-1}(\overline{d_2D_{\lambda-2}}+\overline{d_1D_{\lambda-1}})\\ &= (-)^{\lambda-1}\overline{d_1D_{\lambda-1}} = \frac{(-)^{\lambda-1}}{(\lambda-1)\,!}\overline{d^\lambda} = (-)^{\lambda-1}\lambda D_\lambda. \end{aligned}$$

Hence $$d_\lambda - d_{\lambda-1}D_1+d_{\lambda-2}D_2-d_{\lambda-3}D_3+\ldots+(-)^\lambda\lambda D_\lambda = 0 \quad \ldots\ldots\ldots (7),$$

which, with reference to (3), is Newton's rule for the sums of the powers of the roots, and gives

$$\left.\begin{aligned} d_1 &= D_1\\ d_2 &= D_1-2D_2\\ d_3 &= D_1-3D_1D_2+3D_3\\ d_4 &= D_1^4-4D_1^2D_2+2D_2^2+4D_1D_3-4D_4\\ &\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (8),$$

and in the general case $d_\lambda = [\lambda]$.

3. The differential operators corresponding to the other symmetric functions are found in terms of $d_1, d_2, d_3, \ldots$ by a process parallel to the expression of the symmetric functions in terms of the sums of the powers in the ordinary Algebraic Theory. For

$$d_\lambda d_\mu = \left(\frac{d}{dp_\lambda} + p_1 \frac{d}{dp_{\lambda+1}} + p_2 \frac{d}{dp_{\lambda+2}} + \ldots\right)\left(\frac{d}{dp_\mu} + p_1 \frac{d}{dp_{\mu+1}} + p_2 \frac{d}{dp_{\mu+2}} + \ldots\right)$$

$$= \overline{\left(\frac{d}{dp_\lambda} + p_1 \frac{d}{dp_{\lambda+1}} + \ldots\right)\left(\frac{d}{dp_\mu} + p_1 \frac{d}{dp_{\mu+1}} + \ldots\right)} + \frac{d}{dp_{\lambda+\mu}} + p_1 \frac{d}{dp_{\lambda+\mu+1}} + \ldots$$

Or, after abbreviation and transposition,

$$\overline{d_\lambda d_\mu} = d_\lambda d_\mu - d_{\lambda+\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9).$$

Comparing this with the well-known Algebraic formula

$$\Sigma \alpha^\lambda \beta^\mu = S_\lambda S_\mu - S_{\lambda+\mu},$$

we see that $\overline{d_\lambda d_\mu} = [\lambda . \mu].$

All other formulæ of this class are deducible from (9); for example,

$$d_\lambda d_\mu d_\nu = d_\lambda (\overline{d_\mu d_\nu} + d_{\mu+\nu})$$

$$= \overline{d_\lambda d_\mu d_\nu} + \overline{d_{\lambda+\mu} d_\nu} + \overline{d_\mu d_{\nu+\lambda}} + \overline{d_\lambda d_{\mu+\nu}} + d_{\lambda+\mu+\nu},$$

which, after further reduction of the three middle terms on the right by means of (9), becomes

$$\overline{d_\lambda d_\mu d_\nu} = d_\lambda d_\mu d_\nu - d_\lambda d_{\mu+\nu} - d_\mu d_{\nu+\lambda} - d_\nu d_{\lambda+\mu} + 2d_{\lambda+\mu+\nu} \ldots\ldots \quad (10),$$

showing that $\overline{d_\lambda d_\mu d_\nu} = [\lambda . \mu . \nu].$

But the law $\overline{d_\lambda d_\mu d_\nu \ldots} = [\lambda . \mu . \nu \ldots],$

which is perfectly general when $\lambda, \mu, \nu, \ldots$ are all different, requires modification to meet the case where some of them are equal. This is a necessary consequence of the modification of the Algebraic formula which gives $\Sigma \alpha^\lambda \beta^\mu \gamma^\nu \ldots$ in terms of the sums of the powers, and makes no difference in formulæ such as (9) and (10), which are absolutely correct, even when all the suffixes $\lambda, \mu, \nu, \ldots$ are equal.

Thus, as particular cases of (9) and 10), we have

$$\left.\begin{aligned} \overline{d_\lambda^2} &= d_\lambda^2 - d_{2\lambda} \\ \overline{d_\lambda^3} &= d_\lambda^3 - 3d_\lambda d_{2\lambda} + 2d_{3\lambda} \\ \overline{d_\lambda d_\mu^2} &= d_\lambda d_\mu^2 - d_\lambda d_{2\mu} - 2d_\mu d_{\lambda+\mu} + 2d_{\lambda+2\mu} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (11),$$

which, when compared with the corresponding Algebraic formulæ

$$2!\, \Sigma \alpha^\lambda \beta^\lambda = S_\lambda^2 - S_{2\lambda},$$

$$3!\, \Sigma \alpha^\lambda \beta^\lambda \gamma^\lambda = S_\lambda^3 - 3S_\lambda S_{2\lambda} + 2S_{3\lambda},$$

$$2!\, \Sigma \alpha^\lambda \beta^\mu \gamma^\mu = S_\lambda S_\mu^2 - S_\lambda S_{2\mu} - 2S_\mu S_{\lambda+\mu} + 2S_{\lambda+2\mu},$$

give the identities

$$\overline{d_\lambda^2} \div 2! = [\lambda^2], \quad \overline{d_\lambda^3} \div 3! = [\lambda^3], \quad \overline{d_\lambda\, d_\mu^2} \div 2! = [\lambda \,.\, \mu^2].$$

And, from these considerations, it follows in general that

$$\overline{d_\lambda^l\, d_\mu^m\, d_\nu^n \ldots} \div l!\, m!\, n! \ldots = [\lambda^l \,.\, \mu^m \,.\, \nu^n \ldots] \ldots\ldots\ldots (12).$$

4. Every known symmetric function formula now gives a relation between the operators, and *vice versâ*. Thus formulæ (6) and (7) (*Proceedings*, Vol. xiii., p. 81) give immediately

$$\overline{d_2^m} \div m! = D_m^2 - 2D_{m-1}D_{m+1} + 2D_{m-2}D_{m+2} - \ldots + (-)^m 2D_{2m},$$

$$\overline{d_2^m\, d_1} \div m! = D_m D_{m+1} - 3D_{m-1}D_{m+2} + 5D_{m-2}D_{m+3} - \ldots$$
$$\ldots + (-)^m (2m+1) D_{2m+1},$$

and if, in (6) of the present paper, we put $\lambda = 1$, $\kappa = n-1$, we have

$$d_1 D_{n-1} = \overline{d_2 D_{n-2}} + \overline{d_1 D_{n-1}}$$

the last term of which is nD_n, since $D_\kappa = \overline{d_1^\kappa} \div \kappa!$.

Hence $$\overline{d_2\, d_1^{n-2}} \div (n-2)! = D_1 D_{n-1} - nD_n,$$

which gives the Algebraic formula

$$(2 \,.\, 1^{n-2}) = p_1 p_{n-1} - np_n.$$

The general formula (6) gives, when interpreted,

$$p_\kappa \Sigma a^\lambda = (\lambda+1 \,.\, 1^{\kappa-1}) + (\lambda \,.\, 1^\kappa) \ldots\ldots\ldots\ldots\ldots (13).$$

If now in (13) we make $\kappa + \lambda = n = \text{const.}$ and $(\lambda \,.\, 1^{n-\lambda}) = u_\lambda$, we have $$u_{\lambda+1} + u_\lambda = p_{n-\lambda} \Sigma a^\lambda.$$

Whence, the value of u_2 being known,

$$(2 \,.\, 1^{n-2}) = u_2 = p_1 p_{n-1} - np_n,$$

$$(3 \,.\, 1^{n-3}) = u_3 = p_{n-2}(p_1^2 - 2p_2) - p_{n-1}p_1 + np_n,$$

$$(4 \,.\, 1^{n-4}) = u_4 = p_{n-3}(p_1^3 - 3p_1p_2 + 3p_3) - p_{n-2}(p_1^2 - 2p_2) + p_{n-1}p_1 - np_n,$$

... ...

and generally

$$u_\lambda = p_{n-\lambda+1}\Sigma a^{\lambda-1} - p_{n-\lambda+2}\Sigma a^{\lambda-2} + \ldots + (-)^{\lambda+1} np_n \ldots\ldots\ldots (14).$$

This includes Newton's series, giving

$$\Sigma a^n = p_1 \Sigma a^{n-1} - p_2 \Sigma a^{n-2} + p_3 \Sigma a^{n-3} - \&c.,$$
$$\Sigma a^{n-1}\beta = p_2 \Sigma a^{n-2} - p_3 \Sigma a^{n-3} + p_4 \Sigma a^{n-4} - \&c.,$$
$$\Sigma a^{n-2}\beta\gamma = p_3 \Sigma a^{n-3} - p_4 \Sigma a^{n-4} + p_5 \Sigma a^{n-5} - \&c.,$$

... ...

the last term in every case being $\pm np_n$.

5. If in (8) we consider $d_1, d_2, d_3 \ldots$ as operating on $\phi(p_1, p_2, p_3, \ldots)$, and their equivalents

$$D_1,\ D_1^2-2D_2,\ D_1^3-3D_1D_2+3D_3 \ldots \text{ on } (\lambda^l . \mu^m . \nu^n \ldots),$$

we obtain a set of linear differential equations of the first order, all of them satisfied by ϕ.

Now, recalling the law of operation of D on ( ), viz.,
$D_\kappa(\lambda^l . \mu^m . \nu^n \ldots) = 0$, $D_\lambda(\lambda^l . \mu^m . \nu^n \ldots) = (\lambda^{l-1} . \mu^m . \nu^n \ldots)$, $D_\lambda(\lambda) = 1$, it is clear that any number less than the weight of ϕ, say κ, none of whose partitions are contained in $(\lambda^l . \mu^m . \nu^n \ldots)$, corresponds to a differential equation of the form $d_\kappa \phi = 0$; and, whenever a sufficient number of such equations can be found, we are able to calculate the value of ϕ without reference to symmetric functions of inferior weight. A case in point is Σa^n, where the differential equations are

$$d_1, d_2, d_3 \ldots d_{n-1}\ \phi = 0,$$

which, since they are of the first order, are more convenient to use than $D_1, D_2 . D_3 \ldots D_{n-1}\ \phi = 0$.

If $\phi = (3^4 . 1)$, it is easily seen that $d_2\phi = 0$, $d_5\phi = 0$, $d_8\phi = 0$, $d_{11}\phi = 0$, and these equations are, in this case, more than sufficient to completely determine ϕ; in fact, without using $d_8\phi = 0$, we find

$$\begin{aligned}(3^4 . 1) = {} & p_4^2 p_5 - 2p_3p_5^2 - p_3p_4p_6 + 5p_2p_5p_6 - 5p_1p_6^2 + 4p_3^2p_7 - 7p_2p_4p_7 \\ & + 2p_1p_5p_7 + 8p_6p_7 - 4p_2p_3p_8 + 11p_1p_4p_8 - 13p_5p_8 + 7p_2^2p_9 \\ & - 10p_1p_3p_9 - p_4p_9 - 7p_1p_2p_{10} + 17p_3p_{10} + 10p_1^2p_{11} - 13p_2p_{11} \\ & - 10p_1p_{12} + 13p_{13}.\end{aligned}$$

Whence

$$\begin{aligned}d_1\phi = (3^4) = {} & p_4^3 - 3p_3p_4p_5 + 3p_2p_5^2 + 3p_3^2p_6 - 3p_2p_4p_6 - 3p_1p_5p_6 + 3p_6^2 \\ & - 3p_2p_3p_7 + 6p_1p_4p_7 - 3p_5p_7 + 3p_2^2p_8 - 3p_1p_3p_8 - 3p_4p_8 \\ & - 3p_1p_2p_9 + 6p_3p_9 + 3p_1^2p_{10} - 3p_2p_{10} - 3p_1p_{11} + 3p_{12}.\end{aligned}$$

Any of the other differential equations, $d_3\phi = 3(3^3 . 1)$, $d_4\phi = 4(3^3)$, &c., may be used to verify the value of ϕ with the help of a table of symmetric functions.

It may be noticed that, if by any method a table of symmetric functions of weight n has been calculated, the complete table of weight $n-1$ may be deduced from it by means of the operator d_1.

If $\phi = (5 . 4 . 3 . 2 . 1)$, the differential equations are found thus:

$$\begin{aligned}d_1\phi &= D_1(5 . 4 . 3 . 2 . 1) = (5 . 4 . 3 . 2), \\ d_2\phi &= (D_1^2 - 2D_2)(5 . 4 . 3 . 2 . 1) \\ &= D_1^2(5 . 4 . 3 . 2 . 1) - 2(5 . 4 . 3 . 1) = -2(5 . 4 . 3 . 1), \\ d_3\phi &= (D_1^3 - 3D_1D_2 + 3D_3)(5 . 4 . 3 . 2 . 1) \\ &= (-3D_1D_2 + 3D_3)(5 . 4 . 3 . 2 . 1) \\ &= -3(5 . 4 . 3) + 3(5 . 4 . 2 . 1), \\ d_4\phi &= (4D_1D_3 - 4D_4)(5 . 4 . 3 . 2 . 1) = 4(5 . 4 . 2) - 4(5 . 3 . 2 . 1), \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\end{aligned}$$

where, since partitions of all numbers up to 15 are contained in 5.4.3.2.1, there is no equation of the form $d_\kappa \phi = 0$. The differential equations of the second order, $d_1 d_1 \phi = 0$, $d_2 d_2 \phi = 0$, $d_2 d_{11} \phi = 0$, $d_1 d_{13} \phi = 0$, may in this case be utilized; and a method will be given by which any symmetric function whatever may be calculated independently, assuming only the value of the coefficient of the general term of Σa^n and the laws of combination of the operators.

6. The weight of an operator is the quantity by which it reduces the weight of its subject. Thus the weight of either d_λ or D_λ is λ, and so for every operator of the present paper; $\dfrac{d^m}{dp_\lambda dp_\mu dp_\nu \ldots}$ is of order m and weight $\lambda+\mu+\nu+\ldots$, and any operator of the form $\overline{d_\lambda d_\mu d_\nu \ldots}$ is of order m and weight $\lambda+\mu+\nu+\ldots$, where m is the number of the suffixes.

In the expanded value of any operator those differential coefficients may be rejected as useless which are either of higher order than the degree of the subject, or of higher weight than the weight of the subject.

Thus $d_\lambda d_\mu = \overline{d_\lambda d_\mu} + d_{\lambda+\mu} = \overline{\left(\dfrac{d}{dp_\lambda} + p_1 \dfrac{d}{dp_{\lambda+1}} + \ldots\right)\left(\dfrac{d}{dp_\mu} + p_1 \dfrac{d}{dp_{\mu+1}} + \ldots\right)} + \dfrac{d}{dp_{\lambda+\mu}} + p_1 \dfrac{d}{dp_{\lambda+\mu+1}} + \ldots,$

and if the subject be of weight $\lambda+\mu+1$, suppose

$$\phi = A p_{\lambda+\mu+1} + B p_1 p_{\lambda+\mu} + C p_\lambda p_{\mu+1} + D p_{\lambda+1} p_\mu + E p_1 p_\lambda p_\mu + \text{other terms},$$

then

$$d_\lambda d_\mu \phi = \left(\frac{d^2}{dp_\lambda dp_\mu} + p_1 \frac{d^2}{dp_{\lambda+1} dp_\mu} + p_1 \frac{d^2}{dp_\lambda dp_{\mu+1}} + \frac{d}{dp_{\lambda-\mu}} + p_1 \frac{d}{dp_{\lambda+\mu+1}}\right)\phi$$
$$= p_1 (E + D + C + B + A).$$

The operator d_n, when performed on a subject of weight n, reduces to $\dfrac{d}{dp_n}$, and any operator of the form $\overline{d_\lambda d_\mu d_\nu \ldots}$ performed on a subject whose weight is equal to its own, reduces to $\dfrac{d}{dp_\lambda} \cdot \dfrac{d}{dp_\mu} \cdot \dfrac{d}{dp_\nu} \ldots$, since none but the over-weighted differential coefficients are affected with multipliers containing $p_1, p_2, p_3, \ldots$

If $\phi = \Sigma a^n$, and if in (9) and (10) respectively we put $\lambda+\mu = n$ and $\lambda+\mu+\nu = n$, we have, since in this case $d_\lambda = 0$, $d_\mu = 0$, $d_\nu = 0$,

$$\frac{d^2\phi}{dp_\lambda dp_{n-\lambda}} = -\frac{d\phi}{dp_n}$$
$$\frac{d^3\phi}{dp_\lambda dp_\mu dp_{n-\lambda-\mu}} = 2\frac{d\phi}{dp_n}.$$

And, by an easy extension,

$$\frac{d^m \phi}{dp_\lambda dp_\mu dp_\nu \dots} = (-)^{m-1}(m-1)! \frac{d\phi}{dp_n} \quad \dots\dots\dots\dots\dots(15),$$

where $\phi = \Sigma a^n$, and $\lambda+\mu+\nu+\dots = n$, the m suffixes $\lambda, \mu, \nu \dots$ being not necessarily all unequal.

For, by the nature of the operation d_κ, we have universally

$$d_\kappa . \overline{d_\lambda d_\mu d_\nu \dots} = \overline{d_\kappa d_\lambda d_\mu d_\nu \dots} + \overline{d_{\lambda+\kappa} d_\mu d_\nu \dots} + \overline{d_\lambda d_{\mu+\kappa} d_\nu \dots} + \overline{d_\lambda d_\mu d_{\nu+\kappa} \dots} + \&c.$$
$$\dots\dots\dots\dots(16),$$

where, if $d_\kappa = 0$, which happens when the subject is $\phi = \Sigma a^n$, or if $\overline{d_\lambda d_\mu d_\nu \dots} = 0$, or in some other cases, the left-hand side vanishes; and if, further, $\kappa+\lambda+\mu+\nu+\dots = n$, the weight of the subject, (16) reduces to

$$\frac{d^{m+1}}{dp_\kappa dp_\lambda dp_\mu dp_\nu \dots} + \frac{d^m}{dp_{\lambda+\kappa} dp_\mu dp_\nu \dots} + \frac{d^m}{dp_\lambda dp_{\mu+\kappa} dp_\nu \dots} + \frac{d^m}{dp_\lambda dp_\mu dp_{\nu+\kappa} \dots} + \dots = 0$$
$$\dots\dots\dots\dots(17),$$

where it is not necessary that the suffixes should be all unequal.

Now, if (15) holds for differentials of the m^{th} order, each term of the m^{th} order in (17) is equal to $(-)^{m-1}(m-1)! \frac{d}{dp_n}$, and, since there are m of them, the term of the $(m+1)^{\text{th}}$ order is equal to $(-)^m m! \frac{d}{dp_n}$; hence (15), which has been seen to hold in the cases $m=1$ and $m=2$, is true for all positive integral values of m.

From (15) and the known value of the coefficient of p_n in Σa^n, viz., $(-)^{n+1} n$, it follows at once that the general term of Σa^n is

$$\frac{(-)^{r+n}(r-1)!\, n}{a!\, b!\, c! \dots l!} p_1^a p_2^b p_3^c \dots p_n^l \quad \dots\dots\dots\dots(18),$$

where $a+b+c+\dots+l = r$, and the indices $a, b, c \dots l$ are the positive integral, including zero, solutions of the equation

$$a+2b+3c+\dots+nl = n.$$

Hence

$$\begin{aligned}
\Sigma a^n = {} & p_1^n - n p_1^{n-2} p_2 + n p_1^{n-3} p_3 - n p_1^{n-4} \left\{ p_4 - (n-3) \frac{p_2^2}{2!} \right\} \\
& + n p_1^{n-5} \{ p_5 - (n-4) p_2 p_3 \} \\
& - n p_1^{n-6} \left\{ p_6 - (n-5) \left(p_2 p_4 + \frac{p_3^2}{2!} \right) + (n-4)(n-5) \frac{p_2^3}{3!} \right\} \\
& + n p_1^{n-7} \left\{ p_7 - (n-6)(p_2 p_5 + p_3 p_4) + (n-5)(n-6) \frac{p_2^2 p_3}{2!} \right\} \\
& - n p_1^{n-8} \left\{ p_8 - (n-7) \left(p_2 p_6 + p_3 p_5 + \frac{p_4^2}{2!} \right) \right. \\
& \left. + (n-6)(n-7) \left(\frac{p_2^2 p_4}{2!} + \frac{p_2 p_3^2}{2!} \right) - (n-5)(n-6)(n-7) \frac{p_2^4}{4!} \right\}
\end{aligned}$$

$$+np_1^{n-9}\left\{p_9-(n-8)(p_2p_7+p_3p_6+p_4p_5)\right.$$
$$+(n-7)(n-8)\left(\frac{p_2^2p_5}{2!}+p_2p_3p_4+\frac{p_3^3}{3!}\right)-(n-6)(n-7)(n-8)\left.\frac{p_2^3p_3}{3!}\right\}$$
$$-np_1^{n-10}\left\{p_{10}-(n-9)\left(p_2p_8+p_3p_7+p_4p_6+\frac{p_5^2}{2!}\right)\right.$$
$$+(n-8)(n-9)\left(\frac{p_2^2p_6}{2!}+p_2p_3p_5+\frac{p_2p_4^2}{2!}+\frac{p_3^2p_4}{2!}\right)$$
$$-(n-7)(n-8)(n-9)\left(\frac{p_2^3p_4}{3!}+\frac{p_2^2p_3^2}{2!\ 2!}\right)$$
$$\left.+(n-6)(n-7)(n-8)(n-9)\frac{p_2^5}{5!}\right\}+\&c. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19).$$

7. Formula (12), combined with the principle of rejecting overweighted differential coefficients, furnishes a simple proof of a law of symmetry, discovered by Prof. Cayley in 1856, but given without proof at the end of the tables in Salmon's *Higher Algebra*.

For, if $(\lambda^l.\mu^m.\nu^n\ldots)=\ldots+Ap_\lambda^{l'}p_\mu^{m'}p_\nu^{n'}\ldots+\ldots$

and $(\lambda^{l'}.\mu^{m'}.\nu^{m'}\ldots)=\ldots+A'p_\lambda^{l}p_\mu^{m}p_\nu^{n}\ldots+\ldots$ } ..... (*a*),

the first equation of (*a*) combined with (12) gives

$$\overline{d_\lambda^l d_\mu^m d_\nu^n\ldots}\div l!\,m!\,n!\ldots=\ldots+AD_\lambda^{l'}D_\mu^{m'}D_\nu^{n'}\ldots+\ldots \quad \ldots(a'),$$

since each side of (*a'*) is equal to

$$[\lambda^l.\mu^m.\nu^n\ldots].$$

Now, using each side of (*a'*) as an operator on the opposite side of the second equation of (*a*), since

$$(\overline{d_\lambda^l d_\mu^m d_\nu^n\ldots}\div l!\,m!\,n!\ldots)\,p_\lambda^l p_\mu^m p_\nu^n\ldots$$
$$=\left(\frac{d}{dp_\lambda}\right)^l\left(\frac{d}{dp_\mu}\right)^m\left(\frac{d}{dp_\nu}\right)^n\ldots\frac{p_\lambda^l p_\mu^m p_\nu^n\ldots}{l!\,m!\,n!\ldots}=1,$$

and $$D_{\lambda'}^{l'}D_{\mu'}^{m'}D_{\nu'}^{n'}\ldots(\lambda^{l'}.\mu^{m'}.\nu^{m'}\ldots)=1,$$

and no other terms survive the operation, we have $A=A'$, which is the first part of the law of symmetry.

If, moreover, $p_\lambda^l p_\mu^m p_\nu^n\ldots=\ldots+B(\lambda^{l'}.\mu^{m'}.\nu^{m'}\ldots)+\ldots$

and $p_{\lambda'}^{l'}p_{\mu'}^{m'}p_{\nu'}^{n'}\ldots=\ldots+B'(\lambda^l.\mu^m.\nu^n\ldots)+\ldots$ } ..... (*b*),

the second equation of (*b*), combined with (12), gives, as before,

$$D_{\lambda'}^{l'}D_{\mu'}^{m'}D_{\nu'}^{n'}\ldots=\ldots+B'\,\overline{d_\lambda^l d_\mu^m d_\nu^n\ldots}\div l!\,m!\,n!\ldots+\ldots \quad \ldots(b');$$

and, using each side of equation (*b'*) as an operator on the opposite side of the first equation of (*b*), precisely the same reasoning as before gives us $B'=B$, which is the second part of the law of symmetry.

8. In this concluding article a method, of universal application, for calculating symmetric functions, is illustrated by the calculation of $(3^2.2^3.1)$.

If $F(D_1, D_2, D_3, \ldots)$ is a rational integral function, it is manifest, from the nature of the operation D_x, that all terms in F may be rejected except such as are factors of $D_\lambda D_\mu D_\nu \ldots$, where the subject is $(\lambda . \mu . \nu \ldots)$, of the same type; *i.e.*, the $\lambda . \mu . \nu \ldots$ in ( ) corresponding exactly with the suffixes of the D's. Thus, if the subject is $(3^2 . 2^3 . 1)$, all terms are to be rejected except factors of $D_3^2 D_2^3 D_1$.

When $F(D_1, D_2, D_3, \ldots) = d_n$, the known value (18) of the general term of Σa^n enables us to write down at once the terms to be retained. Thus, retaining only factors of $D_3^2 D_2^3 D_1$ in the expressions (8) for $d_1, d_2, d_3, \ldots$, we have

$$d_1 = D_1 \qquad d_7 = -7D_1 D_2^3 + 7D_2^2 D_3 + 7D_1 D_3^2$$
$$d_2 = -2D_2 \qquad d_8 = 24D_1 D_2^2 D_3 - 8D_2 D_3^2$$
$$d_3 = -3D_1 D_2 + 3D_3 \qquad d_9 = -9D_2^3 D_3 - 27D_1 D_2 D_3^2$$
$$d_4 = 2D_2^2 + 4D_1 D_3 \qquad d_{10} = -40D_1 D_2^3 D_3 + 15D_2^2 D_3^2$$
$$d_5 = 5D_1 D_2^2 - 5D_2 D_3 \qquad d_{11} = 66D_1 D_2^2 D_3^2$$
$$d_6 = -2D_2^3 - 12D_1 D_2 D_3 + 3D_3^2 \qquad d_{12} = -24D_2^3 D_3^2$$

$$d_{13} = -130D_1 D_2^3 D_3^2.$$

With the values of $d_1, d_2, d_3, \ldots$ thus obtained we form products of weight w, and of 1, 2, 3, ... m factors, where w is the weight of the function to be calculated and m its degree; rejecting in the process all terms that would formerly have been rejected. When this has been done, there remains, of each product, but a single term of the same type as the function to be calculated, and of these terms only the numerical coefficient is retained. These coefficients and their corresponding products are placed opposite each other in contiguous columns,—in the illustrative example thus:

d_{13}	−130	−130	p_{13}
$d_1 d_{12}$	−24	+106	$p_1 p_{12}$
$d_2 d_{11}$	−132	−2	$p_2 p_{11}$
$d_3 d_{10}$	−165	−35	$p_3 p_{10}$
$d_4 d_9$	−90	+40	$p_4 p_9$
$d_5 d_8$	−160	−30	$p_5 p_8$
$d_6 d_7$	−119	+11	$p_6 p_7$
$d_1 d_1 d_{11}$	0	−80	$p_1^2 p_{11} \div 2!$
$d_1 d_2 d_{10}$	−30	+31	$p_1 p_2 p_{10}$
$d_1 d_3 d_9$	−27	−8	$p_1 p_3 p_9$
$d_1 d_4 d_8$	−16	−2	$p_1 p_4 p_8$
$d_1 d_5 d_7$	−35	+8	$p_1 p_5 p_7$
$d_1 d_6 d_6$	−12	−10	$p_1 p_6^2 \div 2!$
$d_2 d_2 d_9$	−108	−14	$p_2^2 p_9 \div 2!$
$d_2 d_3 d_8$	−192	+5	$p_2 p_3 p_8$
$d_2 d_4 d_7$	−84	−3	$p_2 p_4 p_7$
$d_2 d_5 d_6$	−150	+1	$p_2 p_5 p_6$
$d_3 d_3 d_7$	−189	0	
$d_3 d_4 d_6$	−114	0	
$d_3 d_5 d_5$	−225	0	
$d_4 d_4 d_5$	−80	0	

The numbers in the third column are the coefficients of terms whose literal part is given in the fourth, and when these numbers

have been found, the calculation is completed. This may be effected in two distinct ways.

First, considering the operative character of the symbols, if $\phi = (3^2 . 2^3 . 1)$, we have

$$D_1 D_2^3 D_3^2 (3^2 . 2^3 . 1) = 1,$$

and $$d_{13} = -130 D_1 D_2^3 D_3^2, \quad d_1 d_{12} = -24 D_1 D_2^3 D_3^2, \text{ \&c.},$$

giving $d_{13}\phi = -130$, $d_1 d_{12}\phi = -24$, $d_2 d_{11}\phi = -132$, &c.

Whence

$$\frac{d\phi}{dp_{13}} = -130, \quad \frac{d^2\phi}{dp_1 dp_{12}} + \frac{d\phi}{dp_{13}} = -24, \quad \frac{d^2\phi}{dp_2 dp_{11}} + \frac{d\phi}{dp_{13}} = -132, \text{ \&c.},$$

and therefore $$\frac{d^2\phi}{dp_1 dp_{12}} = 106, \quad \frac{d^2\phi}{dp_2 dp_{11}} = -2, \text{ \&c.}$$

So also $$d_1 d_1 d_{11}\phi = 0$$

$$= \left(\frac{d}{dp_1} + p_1 \frac{d}{dp_2} + \ldots\right)\left(\frac{d}{dp_1} + p_1 \frac{d}{dp_2} + \ldots\right)\left(\frac{d}{dp_{11}} + p_1 \frac{d}{dp_{12}} + p_2 \frac{d}{dp_{13}}\right)\phi,$$

which gives, after expanding and rejecting the over-weighted differential coefficients, as in Art. 6,

$$\frac{d^3\phi}{dp_1^2 dp_{11}} + 2\frac{d^2\phi}{dp_1 dp_{12}} + \frac{d^2\phi}{dp_2 dp_{11}} + \frac{d\phi}{dp_{13}} = 0,$$

in which, substituting the values of the three last terms found above, we have finally $$\frac{d^3\phi}{dp_1^2 dp_{11}} = -80.$$

In this way any of the numbers in the third column may be found, but the coefficients of terms of a lower degree must be found before those of terms of the next higher degree.

Second, considering the symbols as symmetric functions of (3), viz., $d_\lambda = \Sigma a^\lambda = [\lambda]$, &c., using the Algebraic formulæ corresponding to (9) and (10), and others of the same kind, we have

$$\text{co. } p_1 p_2^3 p_3^2 \text{ in } \Sigma a^\lambda \beta^\mu = \text{co. } p_1 p_2^3 p_3^2 \text{ in } (S_\lambda S_\mu - S_{\lambda+\mu})$$
$$= \text{co. } D_1 D_2^3 D_3^2 \text{ in } (d_\lambda d_\mu - d_{\lambda+\mu}).$$

Thus, if

$$\lambda = 10, \quad \mu = 3, \quad \text{co. } p_1 p_2^3 p_3^2 \text{ in } \Sigma a^{10} \beta^3 = -165 + 130 = -35,$$

and therefore, by the law of symmetry,

$$\text{co. } p_8 p_{10} \text{ in } (3^2 . 2^3 . 1) = -35.$$

So also

$$\text{co. } p_1 p_2^3 p_3^2 \text{ in } \Sigma a^\lambda \beta^\mu \gamma^\nu$$
$$= \text{co. } p_1 p_2^3 p_3^2 \text{ in } (S_\lambda S_\mu S_\nu - S_\lambda S_{\mu+\nu} - S_\mu S_{\nu+\lambda} - S_\nu S_{\lambda+\mu} + 2S_{\lambda+\mu+\nu})$$
$$= \text{co. } D_1 D_2^3 D_3^2 \text{ in } (d_\lambda d_\mu d_\nu - d_\lambda d_{\mu+\nu} - d_\mu d_{\nu+\lambda} - d_\nu d_{\lambda+\mu} + 2d_{\lambda+\mu+\nu}),$$

and if $\lambda = 2$, $\mu = 4$, $\nu = 7$, we have

$$\text{co. } p_1 p_2^3 p_3^2 \text{ in } \Sigma \alpha^7 \beta^4 \gamma^2 = -84 + 132 + 90 + 119 - 260 = -3,$$

and therefore, by the law of symmetry,

$$\text{co. } p_2 p_4 p_7 \text{ in } (3^2 . 2^3 . 1) = -3.$$

In this way any of the numbers in the third column may be found, when those in the second are known. The method is in its essence that given in all the text-books for finding symmetric functions from the sums of the powers, but it is simplified by the rejection of superfluous terms and by the application of the law of symmetry.

On a Generalization of the Nine-Points Properties of a Triangle.

By Captain P. A. MacMahon, R.A.

[*Read Feb. 8th*, 1883.]

In the triangle ABC (Fig. p. 130), let O be the centre of the circle ABC, T the orthocentre; through O and T draw the lines OL, TN making angles α and $\pi - \alpha$ respectively with the side BC, meeting that side in the points L and N; again, draw the lines OI, TM making angles $\pi - \alpha$ and α with the same side, meeting it in the points I and M; obtain in a similar manner eight other points, four on each of the other sides: these twelve points lie six and six upon two equal circles of radius $\frac{1}{2}R \operatorname{cosec} \alpha$, R being the radius of the circle ABC.

These two circles also pass each through six other points, corresponding to the points bisecting TA, TB, TC which lie upon the nine-points circle of the triangle.

When $\alpha = \frac{\pi}{2}$, the two circles considered here coalesce into the nine-points circle. Also, as will be seen, the twelve other points mentioned coalesce in this case into three.

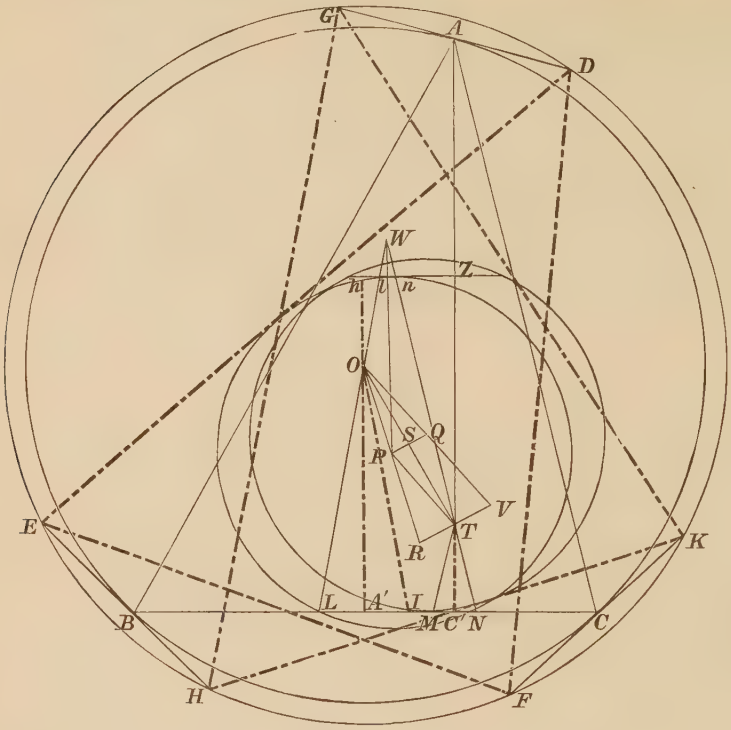
1. Let S be the nine-points centre, and draw SP at right angles to OT, and OP making an angle α with SP. Then P is the centre of the circle passing through the points L and N and the corresponding points on the other sides of the triangle.

Draw OA', TC' perpendiculars to BC, meeting it in the points A and C'.

Join PL, PN, PT, SA', SC'.

Since, in the two triangles LOP, $A'OS$,

$$\text{angle } LOP = \text{angle } A'OS,$$



and
$$\frac{OL}{OP} = \frac{OA'}{OS}.$$

Therefore the triangles are similar, and

$$\frac{LP}{OP} = \frac{A'S}{OS}.$$

But $OP = OS \operatorname{cosec} \alpha$, and $A'S = \frac{1}{2}R$. Therefore

$$PL = \tfrac{1}{2}R \operatorname{cosec} \alpha.$$

In the same manner, from the similarity of the triangles PTN, STC', we find
$$PN = \tfrac{1}{2}R \operatorname{cosec} \alpha.$$

Similarly, if PS be produced to Q, so that $SQ = SP$, it may be shown that an equal circle centred at Q passes through the points M, I, and the four other corresponding points on the other sides of the triangle.

Thus the first part of the proposition is proved.

In the figure, produce OP and OQ to R and V, making

$$QV = PR = OP.$$

With centre O and radius $R \operatorname{cosec} \alpha$, describe the circle $DGEHFK$. At A, B, and C draw tangents DG, EH, FK to the circle ABC.

Then the circle whose centre is at P passes through the middle points of the lines RD, RG, RE, RH, RF, RK, and also touches the inscribed and escribed circles of the triangle DEF, which is the corresponding Feuerbach property.

Also the circle centred at Q passes through the middle points of the lines VD, VG, VE, VH, VF, VK, and touches the inscribed and escribed circles of the triangle GHK.

For, clearly, R is the external centre of similitude of the outer circle, and the circle centred at P; and since

$$RO = 2PO,$$

therefore lines radiating from R to the circumference of the outer circle are bisected by the circumference of the circle centred at P.

Again, the angle EOF = angle BOC.

Therefore the triangles DEF, ABC are similar, and they lie similarly with regard to the points (O, P, R), (O, S, T) respectively.

Therefore the circle centred at P is the nine-points circle of the triangle DEF, and consequently it touches its inscribed and escribed circles.

The like properties of the circle centred at Q, which is the nine-points circle of the triangle GHK, follow from similar reasoning.

Thus, it has been shown that the two circles considered pass each through twelve points; when $\alpha = \frac{\pi}{2}$, the twenty-four points become nine, and the two circles coalesce into the nine-points circle. Mr. Morgan Jenkins has noticed that each circle passes also through six other points, obtained in a different manner, and has kindly added a note on the subject.

It may be observed that the two circles are also the inscribed circles of the two three-cusped hypocycloids which are the envelopes of the two tangents, inclined at angles α and $\pi - \alpha$ to the axis, to a parabola inscribed in the triangle ABC.

The investigation of these envelopes suggested a generalisation of the nine-points properties. (See *Messenger of Mathematics*, January, 1883.)

By the variation of α we obtain a system of twelve-points circles whose envelopes may be found. Taking S as origin, SQ as axis of x, and putting $OS = p$, we have to find the envelope of

$$y^2 + (x - p \cot \alpha)^2 = \tfrac{1}{4} R^2 \operatorname{cosec}^2 \alpha,$$

or $$\tfrac{1}{2}(p^2 - x^2 - y^2) \cos 2\alpha - px \sin 2\alpha + \tfrac{1}{2}(x^2 + y^2) - \tfrac{1}{4} R^2 + \tfrac{1}{2} p^2 = 0.$$

This envelope is

$$\tfrac{1}{4}(p^2-x^2-y^2)^2+p^2x^2=\tfrac{1}{4}(x^2+y^2+p^2-\tfrac{1}{2}R^2)^2,$$

which reduces to $$\frac{4y^2}{R^2}-\frac{4x^2}{4p^2-R^2}=1.$$

Since $$4p^2=R^2(1-8\cos A\cos B\cos C),$$

this represents an hyperbola or an ellipse of semi-axes

$$\tfrac{1}{2}R \text{ and } R\sqrt{(\pm 2\cos A\cos B\cos C)},$$

according as the triangle is obtuse or acute-angled.

If the triangle be right-angled, the envelope consists of two points; viz., the right angle, and the middle point of the hypothenuse.

Note on Captain MacMahon's Paper. *By* Morgan Jenkins, M.A.

Through Z, the middle point of AT, draw $Znlh$, parallel to BC, meeting NT, LO and $A'O$ produced in n, l, and h respectively. Then, because S is the middle point of ZA', the diagonal of the rectangle $ZhA'C'$, $$Sh=SZ=\tfrac{1}{2}R;$$

and, as the triangles $A'OS$, LOP have been proved to be similar, so the triangles ZTS, nTP are similar, and the triangles hOS, lOP are similar;

$$Pn=SZ\operatorname{cosec}\alpha=\tfrac{1}{2}R\operatorname{cosec}\alpha;\quad Pl=Sh\operatorname{cosec}\alpha=\tfrac{1}{2}R\operatorname{cosec}\alpha.$$

Therefore the circle whose centre is at P passes through six other points, viz., l, n, and the four corresponding points; and, in a similar manner, the circle whose centre is at Q passes through six other points.

It may be noticed that the points W, O, P, T are concyclic, because the angle $OPT=2$. angle $OPS=2\alpha$, and, from the triangle WLN, the angle $LWN=\pi-2\alpha$. Hence, since $OP=TP$, WP bisects the angle OWT, and is therefore perpendicular to BC. This also follows from the fact that PLN and WLN are both isosceles triangles.

Thursday, March 8th, 1883.

Prof. HENRICI, F.R.S., President, and subsequently Sir J. COCKLE, F.R.S., V.-P., in the Chair.

Mr. Alfred Lodge, M.A., Fereday Fellow of St. John's College, Oxford, was elected a Member. Major A. Cunningham, R.E., and Mr. H. T. Gerrans were admitted into the Society. Prof. Henrici announced to the Members the loss the Society had sustained by the death of Prof. Henry Smith, F.R.S.

The following communications were made:—

"Calculation of the Hyperbolic Logarithm of π:" Mr. J. W. L. Glaisher, F.R.S.

"On Monge's Mémoire sur la Théorie des Déblais et des Remblais:" Prof. Cayley, F.R.S.

"Remarks on a recent Paper by Prof. Sylvester, F.R.S., in the *American Journal of Mathematics*:" Mr. J. Hammond, M.A.

The following presents were received:—

Cartes-de-visite likenesses of Profs. J. C. Adams, F.R.S., G. G. Stokes, F.R.S., and P. G. Tait, presented by Mr. R. F. Scott; and a cabinet likeness of Dr. Hirst, F.R.S., presented by Dr. Hirst.

"American Journal of Mathematics," Vol. v., No. 2.

"Association for Improvement of Geometrical Teaching." Ninth General Report, January, 1883.

"The Mathematical Magazine, a Journal of Elementary Mathematics," Edited by Artemas Martin, M.A., Vol. i., No. 5 (Jan. 1883); Erie, Pa.

"Tidskrift for Mathematik," udgivet af H. G. Zeuthen, Fjerde Række, Sjette Aargang, Første, ... Sjette Hefte; Kjøbenhavn, 1882.

"Sur un groupe de Théorèmes et Formules de la Géométrie énumérative," par H. G. Zeuthen, (extracted from No. 1 of the *Acta Mathematica*).

"Educational Times" for March.

The following 8vo pamphlets by Mr. J. W. L. Glaisher:—

"Various Papers and Notes that have appeared in the *Quarterly Journal of Mathematics*, and the *Messenger of Mathematics*, during the years 1881 and 1882;" Cambridge, 1882.

"On some Definite Integrals expressible in terms of the First Complete Elliptic Integral and of Gamma Functions" (extracted from the *Proc. Lond. Math. Soc.*, Vol. xii., Nos. 171 and 172).

"On the General Analogy between the Formulæ of singly and doubly Periodic Functions" (from *Brit. Assoc. Report*, 1881).

"Note on certain Symbolic Operators, and their application to the solution of certain Partial Differential Equations" (from *Proc. Lond. Math. Soc.*, Vol. xii., No. 178).

"On a Partial Differential Equation" (from *Brit. Assoc. Report*, 1882).

"Biographical Notice of Henry Parr Hamilton" (from the Monthly Notices of the R. A. S., Vol. xli., No. 4).

"Une Identité Trigonométrique" (read before the French Association, 1880).

"Report on Mathematical Tables" (from *Brit. Assoc. Report*, 1881).

From the Royal Dublin Society :—

"Scientific Proceedings," Vol. iii., Part v., 8vo; Dublin, 1882, August.

"Scientific Transactions," Vol. i. (Series ii.), Parts xv.—xix.; 4to, Dublin, 1882. Also Vol. ii. (Series ii.), Part ii.

"On the Motion of a Projectile in a Resisting Medium," by Prof. A. G. Greenhill, 8vo; Woolwich, 1882. From the Author.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome vii., Jan. 1883.

"Beiblätter zu den Annalen der Physik und Chemie," B. vii., S. ii.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," No. 3, 1883.

"Acta Mathematica," Vol. i., Part ii.

"Atti della R. Accademia dei Lincei," Series iii., Vol. vii., Fasc. 3.

"Bulletin Astronomique et Météorologique de l'Observatoire de Rio de Janeiro," No. 12 (Dec. 1882).

"Crelle," Band 94, Heft 1.

"Memorie della R. Accademia di Scienze, Lettere, ed Arti in Modena," Vol. xx., Parts i. and ii.

Calculation of the Hyperbolic Logarithm of π.

By J. W. L. Glaisher, M.A., F.R.S.

[*Read March 8th*, 1883.]

1. In the Introduction to his *Table des Diviseurs pour tous les nombres du premier Million* (Paris, 1817), Burckhardt explains the application of the table to the calculation of logarithms, taking as his example the hyperbolic logarithm of π.

By means of the table, he finds that $3141593 = 7\times13\times19\times23\times79$, but he remarks that a better approximation to π is afforded by 31415925, which $= 3\times5^2\times373\times1123$. As an approximation which is closer still, he gives $314159265 = 3^2\times5\times7\times127\times7853$.

Adding the logarithms of the factors forming this last product, and subtracting $8\times\log_e 10$, he finds

$$\log_e 3{\cdot}14159265 = 1{\cdot}144\,729\,884\,706\,733\,496\,332\,198\,442\,742\,4,$$

which differs from $\log_e\pi$ only in the ninth decimal place. Putting

$$\frac{\pi}{314159265} = 1+\alpha,$$

he finds $\quad \alpha = 11{\cdot}426\,666\,784\,640\,724\,771\,493\times10^{-31}$,

giving $\quad \log_e\pi = 1{\cdot}144\,729\,885\,849\,400\,174\,143\,427.$

Burckhardt also mentions that we may obtain even better approximations by seeking for a number which shall be nearly equal to π, and

also be the difference of two squares. Thus, for example,

$$(1772454)^2 = 314\,159\,318\,211\,6, \quad \pi = 314\,159\,265\,359\,0,$$

the difference $528526 = (727)^2$ nearly, whence

$$\pi = (1772454)^2 - (727)^2 = (1772454+727)(1772454-727)$$
$$= 1773181 \times 1771727.$$

The first factor being a prime, this formula, he remarks, is not available; but, as the product is not changed sensibly by increasing one of the factors by a small quantity and diminishing the other by the same quantity, he adds $\frac{2}{10}$ to one factor, and subtracts it from the other, and thus obtains the formula

$$17731808 \times 17717272 = 2^8 \times 19 \times 173 \times 229 \times 509 \times 3203$$
$$= 314\,159\,265\,387\,776.$$

2. Wolfram's table gives the hyperbolic logarithms of all numbers up to 2,200, and of all primes (and of many composite numbers also) up to 10,009, to 48 places of decimals. It first appeared in Schulze's *Sammlung logarithmischer... Tafeln*, pp. 190—259 (Berlin, 1778), and was reprinted in Vega's *Thesaurus logarithmorum completus*, pp. 642—684 (Leipzig, 1794), with the addition of six logarithms which had been omitted owing to Wolfram's death. By means of this table, therefore, we can obtain at once, by simple addition, the hyperbolic logarithm of any number which can be resolved into factors none of which exceeds 10,009. The method of calculating a logarithm to which Burckhardt draws attention is a very convenient one, and I was tempted by my father's calculation of the factor tables of the fourth, fifth, and sixth millions (which, in conjunction with Burckhardt's and Dase's tables, afford the means of resolving into factors at once any number not exceeding 9,000,000) to seek for a number very nearly equal to π and having no prime factor exceeding 10,009, and in this manner to calculate the hyperbolic logarithm of π to 48 places of decimals.

3. The two nearest approximations to π, satisfying the required conditions, which I was successful in finding were:—

(i.) 3·141 592 653 584 947 2,

which differs from π by ·000 000 000 004 846 0 ...,

and (ii) 3·141 592 653 535,

which differs from π by ·000 000 000 054 ...;

the resolutions into factors being as follows:—

(i.) $314\,159\,265\,358\,494\,72 = 2^{\ } \times 3 \times 19 \times 269 \times 307 \times 3433 \times 3797$;

(ii.) $314\,159\,265\,353\,5 = 5 \times 7 \times 17^2 \times 19 \times 37 \times 107 \times 4129$.

4. Taking the first resolution (i.), we have

$$\pi = x+h = 3{\cdot}141\ 592\ 653\ 589\ 793\ 238\ \ldots,$$
$$x\ \ = 3{\cdot}141\ 592\ 653\ 584\ 947\ 2,$$

whence $h = {\cdot}(11)\ 4\ 846\ 038\ 462\ 643\ 383\ 279\ 502\ 884$
$197\ 169\ 399\ 375\ 105\ 820\ 974\ 945\ \ldots,$

where the notation (11) indicates that there are eleven ciphers between the decimal point and the first significant figure.

Dividing the value of h by x, and retaining 37 figures of the quotient, we find, true to 48 decimal places,

$$\frac{h}{x} = {\cdot}(11)\ 1\ 542\ 541\ 951\ 488\ 666\ 689\ 372\ 045\ 964\ 447\ 941\ 218.$$

Multiplying this value of $\frac{h}{x}$ by itself, and dividing by 2, we deduce

$$\tfrac{1}{2}\frac{h^2}{x^2} = {\cdot}(23)\ 1\ 189\ 717\ 836\ 051\ 232\ 069\ 032\ 064\,;$$

multiplying the value of $\frac{h^2}{x^2}$ by that of $\frac{h}{x}$, and dividing by 3,

$$\tfrac{1}{3}\frac{h^3}{x^3} = {\cdot}(35)\ 1\ 223\ 459\ 781\ 696\,;$$

and the next term $\tfrac{1}{4}\frac{h^4}{x^4} = {\cdot}(47)\ 1.$

Taking the logarithms from Wolfram's table, it is found that

$$\log_e (x \times 10^{16}) = 37{\cdot}\ 986\ 091\ 373\ 752\ 588\ 576\ 479\ 803$$
$$149\ 331\ 350\ 037\ 309\ 079\ 940\ 128,$$

whence, subtracting $16 \times \log_e 10$,

$$\log_e x = 1{\cdot}\ 144\ 729\ 885\ 847\ 857\ 632\ 191\ 939$$
$$874\ 381\ 522\ 715\ 691\ 456\ 122\ 064.$$

Substituting for $\log_e x$, $\frac{h}{x}$, &c., their values in the formula

$$\log_e \pi = \log_e x + \frac{h}{x} - \tfrac{1}{2}\frac{h^2}{x^2} + \tfrac{1}{3}\frac{h^3}{x^3} - \tfrac{1}{4}\frac{h^4}{x^4} + \&c.,$$

we finally obtain the result

$$\log_e \pi = 1{\cdot}\ 144\ 729\ 885\ 849\ 400\ 174\ 143\ 427$$
$$351\ 353\ 058\ 711\ 647\ 294\ 812\ 913.$$

5. Similarly, in the case of the second resolution (ii.), the values of the same expressions, obtained in the same manner, are :—

$$\pi = x+h = 3{\cdot}\ 141\ 592\ 653\ 589\ 793\ \ldots,$$
$$x\ \ = 3{\cdot}\ 141\ 592\ 653\ 535,$$

$$h = \cdot(10)\ 54\ 793\ 238\ 462\ 643\ 383\ 279\ 502\ 884$$
$$197\ 169\ 399\ 375\ 105\ 820\ 974\ 944\ \ldots,$$

$$\frac{h}{x} = \cdot(10)\ 17\ 441\ 229\ 498\ 989\ 513\ 169\ 562\ 245\ 806\ 364\ 125\ 677,$$

$$\tfrac{1}{2}\frac{h^2}{x^2} = \cdot(21)\ 152\ 098\ 243\ 218\ 210\ 992\ 284\ 118\ 085,$$

$$\tfrac{1}{3}\frac{h^3}{x^3} = \cdot(32)\ 1\ 768\ 520\ 244\ 241\ 295,$$

$$\tfrac{1}{4}\frac{h^4}{x^4} = \cdot(43)\ 23\ 134,$$

$$\log_e x = 1\cdot 144\ 729\ 885\ 831\ 958\ 944\ 644\ 589$$
$$936\ 426\ 712\ 908\ 312\ 970\ 587\ 159,$$

giving $$\log_e \pi = 1\cdot 144\ 729\ 885\ 849\ 400\ 174\ 143\ 427$$
$$351\ 353\ 058\ 711\ 647\ 294\ 812\ 912.$$

6. The two results differ only by a unit in the forty-eighth place, and, since all the portions of the two calculations are entirely distinct, we may therefore conclude with certainty that to forty-seven places of decimals the value of the hyperbolic logarithm of π is

$$1\cdot 144\ 729\ 885\ 849\ 400\ 174\ 143\ 427$$
$$351\ 353\ 058\ 711\ 647\ 294\ 812\ 91.$$

7. The resolution (ii.) was found without the aid of any definite process; but the resolution (i.) was the result of a systematic series of trials by Burckhardt's method. The first seventeen figures of π were subtracted from $(177245386)^2$, $(177245387)^2$, ... $(177245400)^2$ in succession, and the square nearest to the difference in each case was taken out from Kulik's *Tafeln der Quadrat- und Kubik-Zahlen*, which gives squares and cubes of numbers up to 100,000. The differences of squares thus obtained give resolutions into pairs of factors, each containing nine figures, and these were resolved into their prime factors as far as was found practicable or necessary. In only one case were the factors thus found all included within the range of Wolfram's table; the resolution (i.) was thus obtained, the process being as follows:—The difference between $(177245396)^2$ and the number whose digits are the first seventeen figures of π is 38672988884, the nearest square to which is $3867347344 = (62188)^2$,

and $$(177245396)^2 - (62188)^2 = 177307584 \times 177183208$$
$$= (2^6 \times 3 \times 269 \times 3433) \times (2^3 \times 19 \times 307 \times 3797),$$

which gives (i.).

The following resolution derived from this system may also be noticed:— $$(177245387)^2 - (26017)^2 = 177271404 \times 177219370$$
$$= (2^2 \times 3 \times 47 \times 257 \times 1223) \times (2 \times 5 \times 23 \times 770519);$$

but the last factor 770519 is prime. The product in this case is 314 159 265 358 954 80, which differs from π in the fourteenth figure.

8. I made some attempts in which the larger of the two numbers that were squared contained ten figures; the one which came nearest to success was

$$(1772453853)^2-(86167)^2 = 1772540020\times 1772367686.$$

The first factor $= 2^2\times 5\times 17\times 19\times 89\times 3083,$

and the second $= 2\times 79\times 11217517;$

but I found no factor of 11217517. The product in this case differs from π only in the seventeenth figure.

It may also be remarked that

$$(17724539)^2-(4171)^2 = 17728710\times 17720368$$
$$= (2\times 3\times 5\times 19^2\times 1637)\times(2^4\times 1107523).$$

The product differs from π in the twelfth figure.

9. The different resolutions which seem worthy of notice are collected together in the following list. The number in [ ] denotes the number of figures which agree with those of π; and the two figures following the [ ] are the first two figures of the difference between the given number and the figures of π.

(i.) 314 159 265 358 494 72 $= 2^9\times 3\times 19\times 269\times 307\times 3433\times 3797,$

Difference $=$ [12] 48.

(ii.) 314 159 265 353 5 $= 5\times 7\times 17^2\times 19\times 37\times 107\times 4129,$

Difference $=$ [11] 54;

these are the resolutions by means of which $\log_e\pi$ was calculated in §§ 4 and 5.

(iii.) 314 159 265 387 776 $= 2^8\times 19\times 173\times 229\times 509\times 3203,$

Difference $=$ [10] 28;

this is Burckhardt's resolution (§ 1).

(iv.) 314 159 265 42 $= 2\times 3\times 7^2\times 11\times 13\times 19\times 67\times 587,$

Difference $=$ [10] 61.

(v.) 314 159 265 358 954 80 $= 2^3\times 3\times 5\times 23\times 47\times 257\times 1223\times 770519,$

Difference $=$ [13] 24, (§ 7).

(vi.) 314 159 265 365 280 $= 2^5\times 3\times 5\times 19^2\times 1637\times 1107523,$

Difference $=$ [11] 63, (§ 8).

(vii.) 314 159 265 358 979 372 0 $= 2^3\times 5\times 17\times 19\times 79\times 89\times 3083$ $\times 11217517,$

Difference $=$ [16] 48, (§ 8).

10. In applying the method of calculating logarithms made use of in this note, I have invariably experienced but little difficulty in find-

ing a number having nine or ten figures identical with those of a given number, and admitting of resolution into prime factors less than 10,000; but, judging from the preceding resolutions, it appears that the difficulty is very greatly increased when the number of figures exceeds ten.

On Monge's "Mémoire sur la Théorie des Déblais et des Remblais."

By Prof. Cayley, F.R.S.

[*Read March 8th*, 1883.]

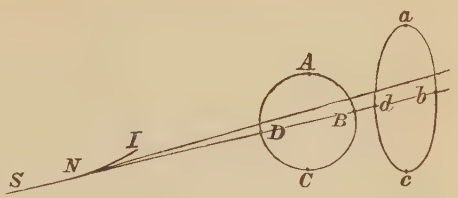
The Memoir referred to, published in the *Mémoires de l'Académie*, 1781, pp. 666 to 704, is a very remarkable one, as well for the problem of earthwork there considered as because the author was led by it to his capital discovery of the curves of curvature of a surface. The problem is, from a given area, called technically the Déblai, to transport the earth to a given equal area, called the Remblai, with the least amount of carriage. Taking the earth to be of uniform infinitesimal thickness over the whole of each area (and therefore of the same thickness for both areas), the problem is a plane one: viz., stating it in a purely geometrical form, the problem is, Given two equal areas, to transfer the elements of the first area to the second area in such wise that the sum of the products of each element into the traversed distance may be a minimum; the route of each element is, of course, a straight line. And we have the corresponding solid problem: Given two equal volumes, to transfer the elements of the first volume to the second volume in such wise that the sum of the products of each element into the traversed distance may be a minimum; the route of each element is, of course, a straight line. The Memoir is divided into two parts: the first relating to the plane problem (and to some variations of it): the second part contains a theorem as to congruences, the general theory of the curvature of surfaces, and finally a solution of the solid problem; in regard to this, I find a difficulty which will be referred to further on.

I have said that Monge gives a theorem as to congruences. This is not stated quite in the best form,—viz., instead of speaking of a singly infinite system of lines, or even of the lines drawn according to a given law from the several points of a *surface*, he speaks of the lines

drawn according to a given law from the several points of a *plane* (but, of course, any congruence whatever of lines can be so represented); and he establishes the theorem that each line of the system is intersected by each of two consecutive lines,—viz., taking (x', y') as the coordinates of the point of intersection of any line with the plane of xy, he obtains, as the condition of intersection with the consecutive line a quadric equation in (dx', dy'). He then considers the normals of a surface (which, as lines drawn according to a given law from any point of a *surface*, require a slightly different analytical investigation), establishes for them the like theorem, and shows moreover that the two directions of passage on the surface to a consecutive point are at right angles to each other; or, what is the same thing, that in the two sets of developable surfaces formed by the intersecting normals, each surface of the one set intersects each surface of the other set in a straight line, and at right angles. He speaks expressly of the lines of greatest and least curvature, and generally establishes the whole theory of the curvature of surfaces in a very complete and satisfactory manner; the particular case of surfaces of the second order is not considered. It may be remarked that, although not explicitly stating it, he must have seen that a congruence of lines is not, *in general*, a system of normals of a surface (that is, the lines of a congruence cannot be, in general, cut at right angles by any surface); he, in fact, assumes (quite correctly, but a proof should have been given) that a congruence of lines for which the two sets of developable surfaces intersect at right angles is a system of normals of a surface.

Reverting to the before-mentioned problem (plane or solid), I remark that this is a problem of minimum *sui generis*. Considering the first area or volume as divided in any manner into infinitesimal elements, we have to divide the second area or volume into corresponding equal elements, in such wise that the sum of the products of each element of the first area or volume into its distance from the corresponding element of the second area or volume may be a minimum; but, for doing this, we have no means of forming the analytical expression of any function which is to be, by the formulæ of the differential calculus or the calculus of variations, made a minimum.

For the plane problem, Monge obtains the solution by means of the very simple consideration that the routes of two elements must not cross each other; in fact, imagine an element A transferred to a, and an equal element B transferred to b: the lines Aa, Bb must not cross each other, for if they did, drawing the two lines Ab and Ba, the sum $Aa+Bb$ would be greater than the sum $Ab+Ba$, contrary to the condition of the minimum. Imagine the areas intersected by



two consecutive lines as shown in the figure: the filament between these two lines may be regarded as a right line; and, assuming that some one element of the filament BD is transferred to a point of bd (that is, so as to coincide with an element of the filament bd), it follows that every other element of BD must be transferred so as to coincide with some other element of bd; and this obviously implies that the filaments BD and bd must be equal. Observe that, this being so, it is immaterial which element of BD is transferred to which element of bd; in whatever way this is done, the sum of the products will be the same.* The two lines may be regarded as the normals of a curve; and the problem thus is, to find a curve such that, drawing the normals thereof to intersect the two areas, then that the filaments BD and bd, cut off by consecutive normals on the two areas respectively, shall be equal. This leads to a differential equation of the second order for the normal curve; one of the constants of integration remains arbitrary, for the normal curve is any one of a system of parallel curves. It is to be observed that the filaments are the increments of the areas BCD and bcd; these increments are equal; a position of the line must be the common tangent Cc of the two areas (this, in fact, constitutes the condition for the determination of one of the arbitrary constants), and for this position the areas are each $= 0$. Hence, in general, the areas must be equal; or the problem is, to find a curve such that any normal thereof cuts off equal areas BCD and bcd.

If instead of the normal curve we consider the curve which is the envelope of the several lines, or, what is the same thing, the locus of the point N, then we could, in like manner, obtain for this curve a differential equation of the first order: the constant of integration would be determined by the condition that Cc is a tangent. The curve in question is, of course, the evolute of the normal curve.

The several lines which intersect the two areas give rise to a finite arc IS of this evolute, and, as remarked by Monge, it is only

---

* The most simple case is, take in the same straight line two equal segments AB, ab; it is immaterial how the elements of AB are transferred to ab, the sum of the products of each element into the traversed distance will be in every case the same. Analytically, if $dx = dx'$, then

$$\int (x'-x)\,dx = \int x'dx' - \int x\,dx,$$

the equation $dx' = dx$ meaning $x' = x + \text{a}$ discontinuous constant. In the actua case of the filament, the formula is, if $rdr = r'dr'$, then

$$\int (r'-r)\,r\,dr = \int r'^2\,dr' - \int r^2\,dr.$$

when this arc *IS* lies (as in the figure) *outside* the two areas, that we have a true minimum.

Passing now to the solid problem, we may imagine a congruence of lines intersecting the two volumes; each line of the congruence is intersected by two consecutive lines, and the lines of the congruence thus form two sets of developable surfaces, each surface of the one set intersecting each surface of the other set. And, considering two consecutive surfaces of the one set, and two consecutive surfaces of the other set, these include between them a filament; and, treating the filament as a right line, it seems to follow (although it is more difficult to present the reasoning in a rigorous form) that, if any one element of the filament *BD* be transferred to any one element of the filament *bd*, then that every other element of the filament *BD* must be transferred to some other element of the filament *bd*; and, this being so, the two filaments must be equal. But Monge goes on to argue that the condition of the minimum further requires that the developable surfaces shall cut at right angles, *and I cannot say that I see this.* He says (pp. 700, 701), "We know already that the routes must be the intersections of two sets of developable surfaces such that each surface of the first set intersects those of the second set in right lines; it remains to be found under what angles these surfaces must cut each other to satisfy the minimum. But it is evident that these angles must be right angles, for with these angles the elementary spaces comprised between four developable surfaces will be greater, and for equal distances the transported mass will be greater; therefore, in the case of a minimum, the routes must be the intersections of two sets of developable surfaces such that each surface of the one set cuts those of the second set in straight lines and at right angles." And, this being so, he infers, and it in fact follows, that the routes are the normals of a surface.

Admitting the conclusion, the problem becomes as follows:— Given two volumes, it is required to find a surface such that, drawing the normals thereof to intersect the two volumes, and considering the filament bounded by the developable surfaces which belong to two consecutive curves of curvature of the one set and those belonging to two consecutive curves of curvature of the other set, the portions cut off on the two volumes respectively may be equal. And we are thus led to a partial differential equation of the second order for determining the equation $z = f(x, y)$ of the required surface. As in the plane problem, it is immaterial how the elements of the one filament are transferred to the other filament.

Thursday, April 12th, 1883.

Professor HENRICI, F.R.S., President, in the Chair.

The Chairman announced that Prof. R. C. Rowe, M.A., had been elected a member of the Council in the room of the late Prof. Smith.

The following communications were made :—

"Equations of the Loci of the Intersections of three Tangent Lines and of three Tangent Planes to any Quadric $u = 0$:" Prof. Wolstenholme, M.A.

"Investigation of the character of the Equilibrium of an incompressible heavy Fluid of variable density :" Lord Rayleigh, F.R.S.

"On the Normal Integrals connected with Abel's Theorem:" Prof. Forsyth, M.A.

"Spherical Functions, Part i.:" Rev. M. M. U. Wilkinson, M.A.

"Calculation of the Equation which determines the Anharmonic Ratios of the Roots of a Quintic:" Prof. M. J. M. Hill, M.A.

"On Simultaneous Differential Equations, with special reference to (1) the Roots of the Fundamental Determinant; (2) the Method of Multipliers:" Mr. E. J. Routh, F.R.S.

The following presents were made :—

"Memoir on the Theta Functions, particularly those of two variables:" by A. R. Forsyth, M.A. (*Phil. Trans.*, Part iii, 1882): from the Author.

"Ellipses," by Henry Perigal, junior, 1855: from the Author.

"Mathematical Questions, with their Solutions, from the *Educational Times*," Vol. xxxviii.; 8vo, London, 1883.

"Proceedings of the Royal Society," Vol. xxxiv., No. 223.

"The Educational Times" for April.

"Johns Hopkins University Circulars," Vol. ii., No. 21.

"Atti della R. Accademia dei Lincei, Serie Terza, Transunti," Vol. vii., Fasc. 4, 5, and 6.

"Sitzungsberichte der Physikalisch-Medicinischen Societät zu Erlangen," 14 Heft., Nov. 1881, bis Aug., 1882; 8vo, Erlangen, 1882.

"Acta Mathematica," &c., Vol. i., Part 3.

"Bulletin de la Société Mathématique de France," Tome xi., No. 1.

"Atti della R. Accademia delle Scienze Fisiche e Matematiche," Vol. ix.; 4to, Naples, 1882.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Anno xxi., 1882; 4to, Naples 1882.

"Annales de l'Observatoire Impérial de Rio de Janeiro," 1st Vol., "Description de l'Observatoire"; 4to, Rio de Janeiro, 1882.

From the "Königlich Sächsische Gesellschaft der Wissenschaften":—

"Abhandlungen," Vol. i. bis xi. (complete); Vol. xii., Nos. 1—4, 8 [Nos. 5, 6, 7 previously received].

"Berichte," 1849—1881.

On Simultaneous Differential Equations, with special reference to—

1. *The Roots of the Fundamental Determinant.*
2. *The Method of Multipliers.*

By Mr. E. J. Routh, F.R.S.

[*Read April 12th*, 1883.]

The propositions in this paper were constructed for the purpose of determining the small oscillations of a system which depends on many coordinates. But, as they are of general application, they are here presented in a form which is purely mathematical. No reference is made to any dynamical principle, and when dynamical terms are used, it is only for the sake of explanation.

We begin by taking the equations of the second order with n dependent variables in their most general forms, though such general forms do not occur in dynamics. Two typical equations are then deduced, and from these all the results given in this paper are derived. These are given in Arts. (5) and (7).

The first step usually taken in solving simultaneous equations is to form a certain determinant. The general form of the solution depends on the nature of the roots of this determinant. Two propositions follow immediately from the typical equations. If three functions, here called A, B, C, be one-signed, it is shown—(1) however general the equations may be, the real roots of the determinant cannot be positive; (2) if the equations be of that simpler character which occurs in dynamics, the real part of every imaginary root is negative. The first proposition has been established by Lagrange and Sir W. Thomson when the equations represent the oscillations of a system about a position of equilibrium. The second is to be found in the author's essay on the "Stability of Motion," but expressed in a different form. It is also given in the last edition of Thomson and Tait's "Natural Philosophy."

The same two typical equations lead immediately to four other theorems which constitute the second part of this paper. When a system of dynamical equations has been solved, it becomes necessary to determine the arbitrary constants in terms of the initial values of the coordinates. Perhaps the equations may have been only partially solved, the rest of the solution depending on some roots of the determinant which cannot be found. In such a case, we may yet require to determine the arbitrary constants which occur in that part of the solution which has been found. The method of Multipliers is used to effect this purpose, and the process is explained in Arts. 11 and 12.

The method of Multipliers is in general use when we express a function by Fourier's rule in a series of sines or cosines, when we expand a function in Laplace's coefficients, and in a great variety of physical problems. The difficulty is, of course, to find the multipliers. In all the cases just alluded to, the rule to find the multipliers is the same as that used in Dynamics when a system oscillates about a position of equilibrium, and the vis viva can be expressed as the sum of the squares of the velocities of the coordinates. The rule in this case is very simple, and is continually used by Lagrange. He also gives, but less distinctly, a rule for the more general case in which the vis viva is any quadric function of the velocities, provided no resistances act on the system.

I do not think that it has been noticed that the multipliers given by Lagrange are not unique. There is another set which may be used when those founded on the vis viva are inconvenient.

Passing over these cases as being sufficiently well known, we shall turn our attention to finding the proper multipliers in the two following cases:—(1) When the equations are such as occur when a system oscillates about a position of equilibrium under forces of resistance which vary as the velocities; (2) when the equations are such as occur when a system oscillates about a state of steady motion, as when a hoop rolls on the ground, and oscillates about the plane of motion. I am not aware that any rule has been given to determine the multipliers in these two cases.

The roots of the Fundamental Determinant.

1. Let there be any number of dependent variables x, y, z, &c., to be found in terms of t, by means of as many differential equations of the second order with constant coefficients. Whatever these equations may be, they may be very conveniently written in the form

$$(A_{11}\delta^2+B_{11}\delta+C_{11})x+\left(\begin{matrix}A_{12}\delta^2+B_{12}\delta+C_{12}\\+D_{12}\delta^2+E_{12}\delta+F_{12}\end{matrix}\right)y+\left(\begin{matrix}A_{13}\delta^2+B_{13}\delta+C_{13}\\+D_{13}\delta^2+E_{13}\delta+F_{13}\end{matrix}\right)z \quad +\&c.=0,$$

$$\left(\begin{matrix}A_{12}\delta^2+B_{12}\delta+C_{12}\\-D_{12}\delta^2-E_{12}\delta-F_{12}\end{matrix}\right)x+(A_{22}\delta^2+B_{22}\delta+C_{22})y+\left(\begin{matrix}A_{23}\delta^2+B_{23}\delta+C_{23}\\+D_{23}\delta^2+E_{23}\delta+F_{23}\end{matrix}\right)z \quad +\&c.=0,$$

$$\left(\begin{matrix}A_{13}\delta^2+B_{13}\delta+C_{13}\\-D_{13}\delta^2-E_{13}\delta-F_{13}\end{matrix}\right)x+\left(\begin{matrix}A_{23}\delta^2+B_{23}\delta+C_{23}\\-D_{23}\delta^2-E_{23}\delta-F_{23}\end{matrix}\right)y+(A_{33}\delta^2+B_{33}\delta+C_{33})z \quad +\&c.=0,$$

$$\&c. \qquad + \qquad \&c. \qquad +\&c.=0,$$

where the symbol δ represents differentiation with regard to t, and the order of suffixes is immaterial, so that $A_{12}=A_{21}$, and so on.

We see here two sets of terms, (1) those which depend on the letters A, B, C, and which by themselves constitute a symmetrical determinant; (2) those which depend on the letters D, E, F, and which by themselves constitute a skew determinant.

2. For the reasons given in Chapter ix. of the fourth edition of the Author's "Rigid Dynamics," we may call the terms which depend on the letter A the *effective forces*, those which depend on the letter B the *forces of resistance*, those on C the *forces of restitution*. It will generally happen that the terms which depend on the letters D and F are absent. The terms which depend on the letter E will occur when we consider the oscillations about a state of motion.* These we shall call the *centrifugal forces*.

If we write A, B, C for the three functions

$$A = \tfrac{1}{2}A_{11}x^2 + A_{12}xy + \tfrac{1}{2}A_{22}y^2 + \ldots\ldots,$$
$$B = \tfrac{1}{2}B_{11}x^2 + B_{12}xy + \tfrac{1}{2}B_{22}y^2 + \ldots\ldots,$$
$$C = \tfrac{1}{2}C_{11}x^2 + C_{12}xy + \tfrac{1}{2}C_{22}y^2 + \ldots\ldots,$$

we see that the terms in the several equations which arise from A, B, C may be written

$$\delta^2\frac{dA}{dx} + \delta\frac{dB}{dx} + \frac{dC}{dx}.$$
$$\text{\&c.} \qquad\qquad \text{\&c.}$$

Hence A, B, C may be called respectively the potentials of the effective forces, the forces of resistance, and the forces of restitution.

3. When we compare the equations of motion with those given by Lagrange for the oscillations about a position of equilibrium, we see that the function A cannot be otherwise than positive. So also these oscillations are stable if the function C be always positive.

Thus, it will frequently occur that the three functions A, B, C, or some of them, are such that they *keep one sign, and do not become zero whatever real quantities* we write for x, y, z, &c. Such functions will be referred to as *one-signed quadrics*.

4. The method of solving the differential equations in Art. (1) is well known. Let m_1, m_2, &c., be the roots of the fundamental determinant, which we need not here write down. Let us suppose that these roots are unequal, the case of equal roots being regarded as a limiting case of unequal roots. The solution may be written thus:—

$$\left.\begin{aligned} x &= x_1 e^{m_1 t} + x_2 e^{m_2 t} + \ldots \\ y &= y_1 e^{m_1 t} + y_2 e^{m_2 t} + \ldots \\ z &= \text{\&c.} \end{aligned}\right\}.$$

* See a Paper contributed in 1875 by the author to the Society (*Proceedings*, Vol. vi., p. 96).

If $x_1' = x_1 m_1$, $y_1' = y_1 m_1$, &c., $x_2' = x_2 m_2$, &c., we have

$$\left.\begin{aligned} \frac{dx}{dt} &= x_1' e^{m_1 t} + x_2' e^{m_2 t} + \dots \\ \frac{dy}{dt} &= y_1' e^{m_1 t} + y_2' e^{m_2 t} + \dots \\ \&c. &= \&c. \end{aligned}\right\}.$$

Here x_1, y_1, z_1, &c. contain as a common factor one constant of integration, x_2, y_2, &c. another constant, and so on. The forms of these constants are not wanted here. It is enough that we should remember that the *coefficients which belong to a real exponential are themselves real.* On the other hand, if m_1, m_2 be a pair of imaginary roots, the coefficients (x_1, x_2), &c., take the form $P \pm Q\sqrt{-1}$.

5. If we substitute the first terms of each of these values of x, y, z, &c., in the equations of Art. (1), we obtain a set of equations which differs from those only in having m_1 written for δ, and x_1, y_1, &c. for x, y, &c.

Multiply these respectively by x_1, y_1, &c., and add the results together; we have

$$(A_{11}x_1^2 + 2A_{12}x_1y_1 + \&c.)\, m_1^2 + (B_{11}x_1^2 + 2B_{12}x_1y_1 + \&c.)\, m_1 + (C_{11}x_1^2 + 2C_{12}x_1y_1 + \&c.) = 0.$$

It should be noticed that the terms which depend on the letters D, E, F have altogether disappeared from this equation.

It should also be noticed that the coefficients of the powers of m are the functions A, B, C with x_1, y_1, &c. written for x, y, &c.

6. Prop. I.—On Real Roots.—We may immediately deduce the three following theorems :—

(1) *If the potentials A, B, C be either zero or one-signed functions, and if all three have the same sign, the fundamental determinant cannot have a real positive root.*

For if m_1 were real, the coefficients x_1, y_1, &c. would be real. We should thus have the sum of three positive quantities equal to zero.

(2) *If there be no forces of resistance, i.e. the term B be absent, and if the potentials A and C be one-signed and have the same sign, the fundamental determinant cannot have a real root, positive or negative.*

(3) *If A, B, C be one-signed functions, but if the sign of B be opposite to that of A and C, the fundamental determinant cannot have a negative root.*

These three propositions, are true, whether there be any terms which depend on the functions D, E, F or not.

7. Exactly as in Art. 5, let us again substitute the first term of each

of the values of x, y, &c. in the equations of motion. But let us now multiply these by x_2, y_2, &c., and add the results. We thus obtain

$$\begin{aligned}&[A_{11}x_1x_2+A_{12}(x_1y_2+x_2y_1)+A_{23}(y_1z_2+y_2z_1)+\&c.]\,m_1^2\\&\quad+[B_{11}x_1x_2+\&c.]\,m_1+[C_{11}x_1x_2+\&c.]\\&=[D_{12}(x_1y_2-x_2y_1)+D_{23}(y_1z_2-y_2z_1)+\&c.]\,m_1^2\\&\quad+[E_{12}(x_1y_2-x_2y_1)+\&c.]\,m_1+[F_{12}(x_1y_2-x_2y_1)+\&c.]\end{aligned}$$

To bring this equation within bounds, we must use some notation to shorten the coefficients. Let us represent the halves of these series by their first terms, omitting suffixes to A, B, &c. We may therefore write the equation in the form

$$A(x_1x_2)\,m_1^2+B(x_1x_2)\,m_1+C(x_1x_2)=D(x_1y_2)\,m_1^2+E(x_1y_2)\,m_1+F(x_1y_2).$$

In the same way we have

$$A(x_1x_2)\,m_2^2+B(x_1x_2)\,m_2+C(x_1x_2)=-D(x_1y_2)\,m_2^2-E(x_1y_2)\,m_2-F(x_1y_2).$$

Also we deduce from these the two equations

$$\left.\begin{aligned}A(x_1x_1)\,m_1^2+B(x_1x_1)\,m_1+C(x_1x_1)&=0\\A(x_2x_2)\,m_2^2+B(x_2x_2)\,m_2+C(x_2x_2)&=0\end{aligned}\right\}.$$

The first of these is the same as that already found in Art. (5).

Here we may notice that the functions $A(xx)$, $B(xx)$, $C(xx)$ are really the same as those we have already more simply denoted by A, B, C. We also notice that $D(x_1y_1)=0$, $E(x_1y_1)=0$, and $F(x_1y_1)=0$.

8. Let us now suppose that there is a pair of imaginary roots in the fundamental determinant of the form $m_1=r+p\sqrt{-1}$, $m_2=r-p\sqrt{-1}$. The values of x, y, &c., given in Art. 4, become

$$\begin{aligned}x&=(x_1+x_2)\,e^{rt}\cos pt+(x_1-x_2)\sqrt{-1}\,e^{rt}\sin pt+\&c.,\\y&=(y_1+y_2)\,e^{rt}\cos pt+(y_1-y_2)\sqrt{-1}\,e^{rt}\sin pt+\&c.,\end{aligned}$$

which may be conveniently abbreviated into

$$\left.\begin{aligned}x&=X_1e^{rt}\cos pt+X_2e^{rt}\sin pt+x_3e^{m_3t}+\dots\\y&=Y_1e^{rt}\cos pt+Y_2e^{rt}\sin pt+y_3e^{m_3t}+\dots\\z&=\&c.\end{aligned}\right\}.$$

If $X_1'=rX_1+pX_2$ and $X_2'=-pX_1+rX_2$, &c.,

$$\left.\begin{aligned}\frac{dx}{dt}&=X_1'e^{rt}\cos pt+X_2'e^{rt}\sin pt+x_3'e^{m_3t}+\dots\\\frac{dy}{dt}&=Y_1'e^{rt}\cos pt+Y_2'e^{rt}\sin pt+y_3'e^{m_3t}+\dots\\\&c.&=\&c.\end{aligned}\right\}.$$

9. Returning now to the two first equations of Art. 7, let us divide them by m_1 and m_2 respectively. If we first add and then subtract the results, we have

$$A(x_1 x_2)\, r + B(x_1 x_2) + C(x_1 x_2)\frac{r}{r^2+p^2} = \left\{ D(x_1 y_2)\, p - F(x_1 y_2)\frac{p}{r^2+p^2} \right\}\sqrt{-1},$$

$$A(x_1 x_2)\, p - C(x_1 x_2)\frac{p}{r^2+p^2} = \left\{ D(x_1 y_2)\, r + E(x_1 y_2) + F(x_1 y_2)\frac{r}{r^2+p^2} \right\}\frac{1}{\sqrt{-1}}.$$

By substitution, we find that

$$\left.\begin{aligned} 4A(x_1 x_2) &= A(X_1 X_1) + A(X_2 X_2) \\ -2D(x_1 y_2)\sqrt{-1} &= D(X_1 Y_2), \end{aligned}\right\}$$

with similar results for the other letters. We also infer from these equations that if *A be a one-signed function*, $A(x_1 x_2)$ *is not only real, but has always the same sign as* A. Similar remarks apply to the functions B and C.

If the functions D, E, F be absent, the two first equations of this Article reduce to

$$\left.\begin{aligned} A(x_1 x_2)\, 2r + B(x_1 x_2) &= 0 \\ -A(x_1 x_2)(r^2+p^2) + C(x_1 x_2) &= 0 \end{aligned}\right\},$$

except when $p = 0$, *i.e.*, except when the roots (which we have supposed imaginary) are real.

10. Prop. II. On Imaginary Roots.—We may immediately deduce the following theorem from the equation of Art. 9.

(1) *Let the fundamental determinant be symmetrical, i.e., let the functions D, E, F be all absent. Let the potentials A and B be one-signed and have the same sign (whether C be a one-signed function or not). Then the real part r of every imaginary root must be negative and not zero. But if the potential B be zero, then the real part of every imaginary root is zero.*

If the potentials A and C be one-signed and have opposite signs, there can be no imaginary roots.

These results follow by simply looking at the two last equations of Art. 9.

(2) *If the terms depending on D and F be absent from the equations, whether the terms depending on E be present or not, and if the three potential functions A, B, C be all one-signed and have the same sign, then the real part r of every imaginary root is negative, and not zero. But if the forces of resistance, i.e. B, be also absent, then the real part of every imaginary root is zero.*

(3) *If the terms depending on D and E* be absent, but not necessarily those depending on F, and if A, B, C be all one-signed and have the

same sign, then the real part r of every imaginary root must be negative, or, if positive, must be less than p.

The Method of Multipliers.

11. We shall now make another application of the equations given in Art. (7).

If we examine the form of the solution of the equations given in Art. (4) or (8), we see that the columns are arranged according to the roots of the fundamental determinant. Each column contains one arbitrary constant which has to be determined from the initial values of the variables x, y, z, &c. If the whole solution be known, we may therefore find the constants by common algebra; though if there be many unknown constants, the process may be very long. But if the whole solution be not known, the processes of common algebra fail.

Thus, suppose we have found only one root of the fundamental determinant, then we know the terms which occur in one column only. The other columns depend on the other roots which have not yet been investigated. We may yet wish to find the value of the constant which occurs in this column in terms of the initial values of the variables. We should thus be able to find the magnitude of any one oscillation without finding the others.

12. To effect this, we use the *method of multipliers*. Our object is to find some multipliers for the equations which express the values of x, y, &c., $\frac{dx}{dt}$, $\frac{dy}{dt}$, &c., such that, on adding together the products, all the columns will disappear, except the one we wish to retain. Supposing this done, we have one equation containing the constant to be found and the initial values of x, y, &c. This equation is sufficient to determine the value of the constant.

13. The two most important problems which occur in Dynamics are those in which we have—

(1) Oscillations about a position of equilibrium, whether with forces of resistance or not.

(2) Oscillations about a state of steady motion.

In the first of these cases, the terms depending on D, E, F in Art. (1) are absent from the equations, and the fundamental determinant is therefore symmetrical. In the second, the terms depending on D, F are absent, but those depending on E are present. The fundamental determinant is therefore unsymmetrical. In general, the forces of resistance, B, are also absent.

We shall now investigate the forms of the multipliers in these two cases.

14. Prop. A.—*To determine the multipliers when the fundamental determinant is symmetrical and the forces of resistance not absent.*

Let $m_1 m_2$ be any two roots of this determinant. Then, by Art. (7),

$$\left.\begin{aligned} A(x_1x_2)\,m_1^2 + B(x_1x_2)\,m_1 + C(x_1x_2) &= 0 \\ A(x_1x_2)\,m_2^2 + B(x_1x_2)\,m_2 + C(x_1x_2) &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1).$$

Eliminating B and C in turn from these equations, we have

$$\left.\begin{aligned} A(x_1x_2)\,m_1m_2 &= C(x_1x_2) \\ -A(x_1x_2)(m_1+m_2) &= B(x_1x_2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(2),$$

except when m_1 and m_2 are the same root.

Either of these equations may be used to find the required multipliers. *We thus find two sets of multipliers.* We shall choose the first equation, as giving the simpler results.

If there be a pair of imaginary roots in the fundamental determinant, say $m_1 = r + p\sqrt{-1}$, $m_2 = r - p\sqrt{-1}$, and if m_3 be any other root, the first of equations (2) gives

$$\left.\begin{aligned} A(x_1x_3)\,(r + p\sqrt{-1})\,m_3 &= C(x_1x_3) \\ A(x_2x_3)\,(r - p\sqrt{-1})\,m_3 &= C(x_2x_3) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(3).$$

Remembering that A and C are linear functions, we see that these give by addition and subtraction

$$\left.\begin{aligned} A(X_1'x_3)\,m_3 &= C(X_1x_3) \\ A(X_2'x_3)\,m_3 &= C(X_2x_3) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4),$$

where X_1X_1', X_2X_2' have the meaning given to them in Art. (8).

The function $A(x_1x_2)$ may obviously be deduced from the potential $A(x_1x_1)$ by the process

$$2A(x_1x_2) = x_2\frac{dA(x_1x_1)}{dx_1} + y_2\frac{dA(x_1x_1)}{dy_1} + \ldots,$$

where of course $A(x_1x_1)$ (see Art. 7) represents the value of $A(xx)$, or A when x_1, y_1, &c. have been written for x, y, &c. The functions B and C may be treated in a similar manner.

We may now immediately deduce the proper multipliers.

Taking the solutions written down in Art. (4) (not reproduced here, to save space), let us multiply the expressions for x, y, &c. by $-\frac{dC}{dx}$, $-\frac{dC}{dy}$, &c., after writing x_1, y_1, &c. in these multipliers for x, y, &c.; also let us multiply the expressions for $\frac{dx}{dt}$, &c. by $\frac{dA}{dx}$, &c.,

after writing x_1', y_1', &c., for x, y, &c., in these multipliers. Finally, let us add the products; then, by virtue of the first of equations (2), the sum of every column except the first is zero.

If we have imaginary roots in the fundamental determinant, we take the solution given in Art. (8). Treating it in the same way, we see by equations (4) that all the columns disappear except the two first. Repeating the process for the second column, we again find that all the columns except the two first disappear.

The rule may be summed up as follows:—

Let the fundamental determinant be symmetrical, and the forces of resistance not absent. Let it be required to separate by the method of multipliers any given column from the others. *The proper multipliers for the coordinates are the values of* $\frac{dC}{dx}$, $\frac{dC}{dy}$, *&c., after we have substituted for* x, y, *&c., in these multipliers the corresponding coefficients in the column we wish to preserve. The proper multipliers for the velocities are the values of* $-\frac{dA}{dx}$, $-\frac{dA}{dy}$, *&c., after we have substituted for* x, y, *&c., in these multipliers the corresponding coefficients in the column of velocities we wish to preserve. Finally, we add the products together.*

When the equations are given, the following rule to find A, B, C will be useful:—*Multiply the equations by* x, y, z, *&c., and add the products, treating the operator* δ *as an algebraical factor. The halves of the coefficients of the powers of* δ *are the functions* A, B, C.

In this way we can find an equation connecting the initial values of the coordinates with the constant which accompanies any one column. Since these initial values are arbitrary, neither side of this equation can wholly vanish unless all the multipliers themselves vanish. Hence the coefficient of the exponential on the right-hand side cannot be zero, except in this one case.

The multipliers cannot all vanish unless the quadric functions C and A also vanish for some *finite values* of the coordinates. In Dynamics the function A is such a function of the coordinates as the vis viva is of the velocities. It is therefore impossible that A could vanish for any finite values of the coordinates.

15. *Example.*—Let us consider the equations

$$\left.\begin{aligned}(\delta^2+\delta+1)\,x+\tfrac{1}{2}\,(\delta-\tfrac{3}{2})\,y=0\\ \tfrac{1}{2}\,(\delta-\tfrac{3}{2})\,x+(\delta^2-\delta+\tfrac{1}{4})\,y=0\end{aligned}\right\}.$$

It is easily seen that the determinant of the solution reduces to

$$m^4-\tfrac{5}{16}=0.$$

We therefore have, if m now stands for $\frac{1}{2}\sqrt[4]{5}$,

$$\left.\begin{aligned} x &= x_1 e^{mt} + x_2 e^{-mt} + X_3 \cos mt + X_4 \sin mt \\ y &= y_1 e^{mt} + y_2 e^{-mt} + Y_3 \cos mt + Y_4 \sin mt \end{aligned}\right\},$$

$$\left.\begin{aligned} \frac{dx}{dt} &= mx_1 e^{mt} - mx_2 e^{-mt} + mX_4 \cos mt - mX_3 \sin mt \\ \frac{dy}{dt} &= my_1 e^{mt} - my_2 e^{-mt} + mY_4 \cos mt - mY_3 \sin mt \end{aligned}\right\}.$$

Also multiplying the equations by x and y, and taking the halves of the coefficients of the powers of δ, we have

$$\left.\begin{aligned} A &= \tfrac{1}{2}(x^2+y^2) \\ C &= \tfrac{1}{2}x^2 - \tfrac{3}{4}xy + \tfrac{1}{8}y^2 \end{aligned}\right\}.$$

Suppose we wish to find the coefficients x_1, y_1 in terms of the initial conditions. Following the rule, we multiply x and y by the differential coefficients of C after we have written x_1, y_1 for x, y in the multipiers. We multiply the velocities by minus the differential coefficients of A, writing in the multipliers mx_1 and my_1 for x and y. Finally, we add the results. Thus we have

$$\left.\begin{aligned} & x(x_1 - \tfrac{3}{4}y_1) + y(-\tfrac{3}{4}x_1 + \tfrac{1}{4}y_1) \\ & -\frac{dx}{dt}mx_1 - \frac{dy}{dt}my_1 \end{aligned}\right\} = \left\{\begin{aligned} & x_1^2 - \tfrac{3}{2}x_1y_1 + \tfrac{1}{4}y_1^2 \\ & -m^2(x_1^2+y_1^2) \end{aligned}\right\} e^{mt}.$$

Putting $t = o$, and giving x, y and their velocities their known initial values, we have one equation to find the constants x_1, y_1. Their ratio,

$$\frac{y_1}{x_1} = -\frac{m^2+m+1}{\frac{1}{2}(m-\frac{3}{2})}, \quad m = \tfrac{1}{2}\sqrt[4]{5},$$

being known from the first equation, we easily find both x_1 and y_1.

If we wish to find the coefficients of the trigonometrical terms, we use two sets of multipliers, because the two imaginary exponentials have become mixed up together in the trigonometrical term; or we may replace them by their imaginary exponentials, and find the coefficients of either by one set of multipliers. Taking the first alternative, one set of multipliers will be respectively

$$X_3 - \tfrac{3}{4}Y_3, \quad -\tfrac{3}{4}X_3 + \tfrac{1}{4}Y_3, \quad -mX_4, \quad -mY_4.$$

The other set will be

$$X_4 - \tfrac{3}{4}Y_4, \quad -\tfrac{3}{4}X_4 + \tfrac{1}{4}Y_4, \quad +mX_3, \quad +mY_3.$$

16. Prop. B.—*To determine the multipliers when the fundamental determinant is symmetrical and the forces of resistance absent.*

This proposition is really included in the last. But as the absence of the function B introduces great simplification, it is worth while to consider this case separately.

Since the forces of resistance are absent, none but even powers of δ enter into the equations. Hence for every root of the fundamental determinant there is another equal in magnitude but contrary in sign. If A and C are one-signed functions, and have same sign, these roots are of the form $\pm p\sqrt{-1}$. Choosing this as the type, we may write the equations of Art. (8) in the form

$$x = X_1 \cos pt + X_2 \sin pt + x_3 e^{m_3 t} + \ldots$$
$$\&c. = \&c.,$$
$$\frac{dx}{dt} = X_1' \cos pt + X_2' \sin pt + x_3' e^{m_3 t} + \ldots$$
$$\&c. = \&c.$$

Here, unless there be equal roots, we have

$$\frac{X_2}{X_1} = \frac{Y_2}{Y_1} = \&c. = \frac{X_1'}{-X_2'} = \frac{Y_1'}{-Y_2'} = \&c. = H,$$

because the ratios of the coefficients of any exponential are expressed by the minors of the fundamental determinant, and these, containing only even powers of m, are the same when the exponents are equal in magnitude but contrary in sign.

Here H will stand for the constant in the second column on the right-hand side of the equations, the constant in the first column being included as a factor in X_1, Y_1, &c., X_2', Y_2', &c.

Since the function B is zero, the equations (2) of Art. (14) reduce to

$$A\,(x_1 x_2) = 0,$$
$$C\,(x_1 x_2) = 0,$$

except when $m_1 = \pm m_2$. For a pair of imaginary roots $m_1 = r + p\sqrt{-1}$, $m_2 = r - p\sqrt{-1}$, combined with a third root m_3, we have (exactly as in that article)

$$\left.\begin{aligned} A\,(X_1 x_3) &= 0 \\ A\,(X_2 x_3) &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} C\,(X_1 x_3) &= 0 \\ C\,(X_2 x_3) &= 0 \end{aligned}\right\}.$$

We may use either the function A or the function C to supply the proper multipliers. *We thus find two sets of multipliers.* Which we should choose depends on the forms of A and C.

If either of these functions contain only the squares of the coordinates, *i.e.*, if it be of the form

$$ax^2 + by^2 + cz^2 + \ldots,$$

it is clear that its differential coefficients will be much simpler than if the terms containing the products of the coordinates were also present. The multipliers are indicated by these differential coefficients, and will therefore also be simpler. That function is therefore to be chosen which has the fewest terms containing the products of the coordinates.

Choosing the function A, we have the following rule to find the multipliers. Let it be required to separate from the others any particular oscillation—say the two columns containing the phase pt. *The proper multipliers for the coordinates* x, y, z, *&c. are the values of* $\frac{dA}{dx}$, $\frac{dA}{dy}$, *&c., after we have substituted for* x, y, *&c. in these multipliers the coefficients of either of the columns containing the phase* pt. *Adding these products, we have one equation from which all the oscillations except the one to be preserved have disappeared. The same multipliers may now be used for the velocities, and thus by a second addition we obtain another equation of the same kind.*

The two equations thus obtained may be written thus:—

$$x\frac{dA(X_1X_1)}{dX_1} + \&c. = 2A(X_1X_1)\{\cos pt + H\sin pt\},$$

$$\frac{dx}{dt}\frac{dA(X_1X_1)}{dX_1} + \&c. = 2A(X_1X_1)\{Hp\cos pt - p\sin pt\}.$$

Putting $t = 0$ either before or after using the multipliers, we have two equations to determine H and the other constant included in X_1, Y_1, &c.

17. Prop. C.—*To determine the multipliers when the forces of resistance are absent but the determinant is skewed by the centrifugal forces.*

Referring to the equations in Art. (1), we omit the terms which depend on the letters B, D, F, but retain those which depend on E (Art. 13). Eliminating x, y, &c., we form the determinant which we have called the fundamental determinant. It is unnecessary to write this determinant, as its form is evident from the merest inspection of the equations.

If in this determinant we write $-\delta$ for δ, the rows of the new determinant are the same as the columns of the old, so that the determinant is unaltered. When expanded, the determinant will contain only even powers of δ, and therefore its roots enter in pairs. We shall therefore take as our standard form of solution, instead of that in Art. (8), the expressions

$$\left.\begin{aligned} x &= X_1\cos pt + X_2\sin pt + x_3e^{m_3t} + \dots \\ y &= Y_1\cos pt + Y_2\sin pt + y_3e^{m_3t} + \dots \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots(1);$$

$$\&c. = \&c.$$

$$\left.\begin{aligned} \frac{dx}{dt} &= X_1'\cos pt + X_2'\sin pt + x_3'e^{m_3t} + \dots \\ \frac{dy}{dt} &= Y_1'\cos pt + Y_2'\sin pt + y_3'e^{m_3t} + \dots \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots(2);$$

$$\&c. = \&c.$$

Here the first two columns represent the most common form of a principal oscillation, and the third column represents any other form. When the centrifugal forces (*i.e.* the terms depending on E) are present, the minors of the fundamental determinant do not contain only even powers of δ. It follows that the coefficients in the second column do not necessarily bear a uniform ratio to those in the first column.

We have by Art. (7) the equations

$$\left.\begin{aligned} A(x_1x_2)\,m_1 + C(x_1x_2)\,\frac{1}{m_1} &= E(x_1y_2) \\ A(x_1x_2)\,m_2 + C(x_1x_2)\,\frac{1}{m_2} &= -E(x_1y_2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Adding these to eliminate the functional symbol E, we find

$$A(x_1x_2)\,m_1m_2 + C(x_1x_2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

except when $m_1 = -m_2$.

We notice also, that by Art. (7),

$$\left.\begin{aligned} A(x_1x_1)\,m_1^2 + C(x_1x_1) &= 0 \\ A(x_2x_2)\,m_2^2 + C(x_2x_2) &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

We might also eliminate the function A or C from the equations (3) instead of the function E, and in each case we may deduce a rule to find the multipliers; but the simplest rule is found by eliminating the function E.

The formulæ (4) resemble that used in Art. (14), and there called (2), except in the sign of A. Proceeding therefore exactly as in that article, we shall deduce the corresponding rule for the multipliers.

Instead of equations (3) of Art. (14), we now have (since $r = 0$)

$$\left.\begin{aligned} A(x_1x_3)\,p\sqrt{-1}m_3 + C(x_1x_3) &= 0 \\ -A(x_2x_3)\,p\sqrt{-1}m_3 + C(x_2x_3) &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Remembering that A and C are linear functions of the letters of any one suffix, these give by addition and subtraction

$$\left.\begin{aligned} A(X_1'x_3)\,m_3 + C(X_1x_3) &= 0 \\ A(X_2'x_3)\,m_3 + C(X_2x_3) &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

where as before $X_1 = x_1 + x_2$, $X_2 = (x_1 - x_2)\sqrt{-1}$,

and $$X_1' = pX_2, \quad X_2' = -pX_1.$$

Also writing $m_1 = p\sqrt{-1}$, $m_2 = -p\sqrt{-1}$ in equations (5), we find by subtraction

$$A(X_1'X_2') + C(X_1X_2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

From these formulæ we now deduce the following rule to find the multipliers.

Let the forces of resistance be absent, and let the fundamental determinant be skewed by the centrifugal forces only. Let it be required to separate any principal oscillation from the others. *Selecting one of the two columns which form the oscillation, the proper multipliers for the coordinates* x, y, *&c. are the values of* $\frac{dC}{dx}$, $\frac{dC}{dy}$, *&c., after we have substituted for* x, y, *&c. in these multipliers the corresponding coefficients in the column selected. The proper multipliers for the velocities are the values of* $\frac{dA}{dx}$, $\frac{dA}{dy}$, *&c., after we have substituted for* x, y, *&c. in these multipliers the coefficients corresponding to these velocities in the column selected. Finally, we add all these products together. We then repeat the process with the coefficients of the other of the two columns which form the oscillation.*

By virtue of equations (5) and (8) it will be found that in each of these processes every column *except one* will disappear from the final summation. But we may notice a curious difference between the columns which contain real exponentials and those which contain trigonometrical expressions. If we operate with the coefficients of one of the former introduced into the multipliers, it is the *companion column which does not disappear;* but if we operate with the coefficients of one of the latter, it is the *column whose coefficients we have used which does not disappear.*

18. *Example.*—Let us consider the equations

$$\left.\begin{aligned}(\delta^2-8)\,x+\sqrt{6}\delta y&=0\\ -\sqrt{6}\,\delta x+(\delta^2+2)\,y&=0\end{aligned}\right\}.$$

It is easily seen that the fundamental determinant reduces to

$$m^4-16=0.$$

Hence we have

$$\left.\begin{aligned}x&=X_1\cos 2t+X_2\sin 2t+x_3e^{2t}+x_4e^{-2t}\\ y&=Y_1\cos 2t+Y_2\sin 2t+y_3e^{2t}+y_4e^{-2t}\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{dx}{dt}&=2X_2\cos 2t-2X_1\sin 2t+2x_3e^{2t}-2x_4e^{-2t}\\ \frac{dy}{dt}&=2Y_2\cos 2t-2Y_1\sin 2t+2y_3e^{2t}-2y_4e^{-2t}\end{aligned}\right\};$$

where

$$\left.\begin{aligned}2x_3&=\sqrt{6}y_3\\ 2x_4&=-\sqrt{6}y_4\end{aligned}\right\},\qquad \left.\begin{aligned}Y_1&=-\sqrt{6}X_2\\ Y_2&=\sqrt{6}X_1\end{aligned}\right\}.$$

Also multiplying the equations (see Art. 14) by x, y, adding and

taking the halves of the coefficients of the powers of δ,

$$\left.\begin{aligned} A &= \tfrac{1}{2}(x^2+y^2) \\ C &= \tfrac{1}{2}(-8x^2+2y^2) \end{aligned}\right\}.$$

The proper multipliers are indicated (Art. 17) by the formula

$$x\frac{dC}{dx}+y\frac{dC}{dy}+\frac{dx}{dt}\frac{dA}{dx}+\frac{dy}{dt}\frac{dA}{dy}.$$

Now $$\frac{dC}{dx}=-8x,\quad \frac{dC}{dy}=2y,\quad \frac{dA}{dx}=x,\quad \frac{dA}{dy}=y.$$

Having chosen the column whose coefficients are to be used in the multipliers, we see by Art. (16) that the proper multiplier for the first equation is minus eight times the coefficient of the column in that equation; the proper multiplier for the second equation is twice the coefficient in that equation; the proper multipliers for the third and fourth equations are the coefficients themselves in those equations.

Suppose first we wish to find x_4 and y_4, then, because the fourth column contains a *real* exponential, we operate with the coefficients of the companion column. The multipliers are therefore

$$\frac{dC}{dx}=-8x_3,\quad \frac{dC}{dy}=2y_3,\quad \frac{dA}{dx}=2x_3,\quad \frac{dA}{dy}=2y_3.$$

Hence we find

$$-8x_3x+2y_3y+2x_3\frac{dx}{dt}+2y_3\frac{dy}{dt}=16y_3y_4e^{-2t};$$

substituting for x_3 in terms of y_3 and putting $t=0$, we find

$$-4\sqrt{6}x+2y+\sqrt{6}\frac{dx}{dt}+2\frac{dy}{dt}=16y_4,$$

which determines y_4 in terms of the initial values of the coordinates and their velocities.

Suppose next we wish to find X_1, X_2. Taking the coefficients of the first column, the multipliers are

$$\frac{dC}{dx}=-8X_1,\quad \frac{dC}{dy}=2Y_1,\quad \frac{dA}{dx}=2X_2,\quad \frac{dA}{dy}=2Y_2.$$

Since these columns contain trigonometrical expressions, we know that when we operate with the coefficients of either column in the multipliers, the other column disappears. Hence, paying no attention to any column except the first, we have

$$-8X_1x+2Y_1y+2X_2\frac{dx}{dt}+2Y_2\frac{dy}{dt}=16(X_1^2+X_2^2)\cos 2t;$$

substituting for Y_1 and Y_2 and putting $t=0$, we find

$$-8X_1x-2\sqrt{6}X_2y+2X_2\frac{dx}{dt}+2\sqrt{6}X_1\frac{dy}{dt}=16(X_1^2+X_2^2).$$

Operating in the same way with the coefficients of the second column, we have

$$-8X_2x+2Y_2y-2X_1\frac{dx}{dt}-2Y_1\frac{dy}{dt}=16\,(X_1^2+X_2^2)\sin 2t;$$

substituting as before, we have

$$-8X_2x+2\sqrt{6}X_1y-2X_1\frac{dx}{dt}+2\sqrt{6}X_2\frac{dy}{dt}=0.$$

These equations determine X_1 and X_2 in terms of the initial values of x, y, and their differential coefficients.

19. Prop. D.—*To consider the effect of equal roots on the rules already given.*

When there are equal roots in the fundamental determinant, we require only some slight modification of our rules. Referring to the general solution exhibited in Art. (4), let us suppose, for example, that there are three roots equal to m_1. Regarding these as the limits of the unequal roots, m_1, m_1+h, m_1+k, we may write that solution in the form

$$x=x_1e^{m_1t}+G\frac{d}{dm_1}(x_1e^{m_1t})+H\frac{d^2}{dm_1^2}(x_1e^{m_1t})+x_4e^{m_4t}+\ldots$$

$$y=y_1e^{m_1t}+G\frac{d}{dm_1}(y_1e^{m_1t})+H\frac{d^2}{dm_1^2}(y_1e^{m_1t})+y_4e^{m_4t}+\ldots$$

&c. = &c.,

$$\frac{dx}{dt}=x_1'e^{m_1t}+\frac{d}{dm_1}(x_1'e^{m_1t})+\frac{d^2}{dm_1^2}(x_1'e^{m_1t})+x_4'e^{m_4t}+\ldots$$

&c. = &c.;

where $x_1'=x_1m_1$, $x_4'=x_4m_4$, &c., and G, H are the two constants in addition to the one included in x_1, y_1, &c.

Two questions now present themselves:—(1) When we use certain multipliers to separate a column which depends on a solitary root such as m_4, will the columns which depend on other equal roots such as m_1 (and therefore contain powers of t as factors) still disappear?

(2) What multipliers must we use to separate the three columns which depend on the three equal roots from the remaining columns?

20. Taking the first of these questions, suppose we wish to separate the fourth column of the equations of Art. (19) from the others. Let us use the same multipliers as if there were no equal roots. It is obvious that, since the three first columns disappear in the general case in which h and k have any values, these columns must also disappear when h and k are indefinitely small. *We therefore infer that any column which depends on a solitary root may be separated by the same rules as before.*

As an example, take the rule given in Prop. A, Art. (14). To separate the fourth column, we multiply the equations by

$$\frac{dC\,(x_4x_4)}{dx_4}\ \&c. - \frac{dA\,(x_4'x_4')}{dx_4'}\ \&c.,$$

and add the products. Since the three first columns must disappear, we have

$$\left.\begin{aligned} &C\,(x_1x_4) - A\,(x_1'x_4') = 0 \\ &C\left(\frac{dx_1}{dm_1}x_4\right) - A\left(\frac{dx_1'}{dm_1}x_4'\right) = 0 \\ &C\left(\frac{d^2x_1}{dm_1^2}x_4\right) - A\left(\frac{d^2x_1'}{dm_1^2}x_4'\right) = 0 \end{aligned}\right\}.$$

The last two of these equations also follow from the first by an evident process.

21. Taking the second question, we wish to find what multipliers will separate the three first columns from the others. But these are supplied by the equations just written down. Since m_4 is any other root, and

$$2C\,(x_1x_4) = \frac{dC}{dx_1}x_4 + \frac{dC}{dy_1}y_4 + \ldots,$$

we have merely to use the multipliers indicated by the coefficients of x_4, y_4, &c. in these equations. The rule may be enunciated as follows:—

Multiply the equations by the proper factors for the first column, treating x_1, y_1, *&c.,* x_1', y_1', *&c. as the coefficients, and add the products. We thus have one of the three required equations. Multiply the equations by the proper factors for the second column as if* $\frac{dx_1}{dm_1}$, $\frac{dy_1}{dm_1}$, *&c.,* $\frac{dx_1'}{dm_1'}$, *&c. were the coefficients, and add the products. We thus obtain the second equation. Lastly, multiply the equation by the proper factors for the third column as if* $\frac{d^2x_1}{dm_1^2}$, *&c.,* $\frac{d^2x_1'}{dm_1^2}$, *&c. were the coefficients, and add the products. We thus have, on the whole, three equations to find the three constants which enter into the three first columns.*

The proper factors just mentioned are those calculated from the coefficients by the rules of Prop. A or Prop. C.

22.* In some cases of equal roots it is known that some of the terms with t as a factor fail to introduce themselves into the solution. The number of constants is then made up by a greater indeterminateness in the coefficients which accompany the exponential. Regarding these equal roots as the limits of unequal roots, as in Art. 20, it

* This and the next article were added after the paper was read.

follows that we can still use the same rules to find the multipliers. We arrange our solution in columns with one constant in each column. Then, using the proper multipliers, as described above, we can separate any solitary root at once. To determine the constants which accompany the equal roots, we shall require as many sets of multipliers as there are columns with that root or its companion root.

23. *Example.*—Let us consider the equations

$$\left.\begin{aligned}(\delta^2-1)\,x+y+z&=0\\ x+(\delta^2-1)\,y+z&=0\\ x+y+(\delta^2-1)\,z&=0\end{aligned}\right\}.$$

It is easily seen that the fundamental determinant reduces to

$$(m^3-2)^2\,(m^2+1)=0.$$

Putting $a=\sqrt{2}$, we write the solution in the form

$$\left.\begin{aligned}x&= Ee^{at} + Ge^{-at} + K\sin t + L\cos t\\ y&= Fe^{at} + He^{-at} + K\sin t + L\cos t\\ z&= -Ee^{at} - Fe^{at} - Ge^{-at} - He^{-at} + K\sin t + L\cos t\end{aligned}\right\},$$

where E, F, G, H, K, L are the six constants to be determined.

Looking at the equations to be solved, we see that the potential functions A and C are given by

$$\left.\begin{aligned}2C&=-x^2-y^2-z^2+2xy+2yz+2zx\\ 2A&=x^2+y^2+z^2\end{aligned}\right\}.$$

Following the rule indicated in Art. 16, we choose the function A to operate with, because this function will supply the simplest multipliers. The proper multipliers will therefore be

$$\frac{dA}{dx}=x,\quad \frac{dA}{dy}=y,\quad \frac{dA}{dz}=z,$$

where we write for x, y, z the coefficients of the column under consideration. The proper multipliers are therefore the coefficients of the columns in succession.

Suppose we wish to find K and L. The coefficients in either of these two columns are all equal. The multipliers are therefore equal. We therefore obtain, by adding the equations and putting $t=0$,

$$x+y+z=3L.$$

Treating the differential coefficients in the same way (Art. 16), we have

$$\delta x+\delta y+\delta z=3K.$$

If we wish to find the four constants E, F, G, H which are all connected with the companion roots $\pm a$, we must find four equations.

According to the rule, the multipliers are the coefficients of the several columns. We thus obtain, when $t = 0$,

$$\left.\begin{aligned} Ex + 0y - Ez &= E\,(2E + 2G + F + H) \\ 0x + Fy - Fz &= F\,(E + G + 2F + 2H) \end{aligned}\right\},$$

$$\left.\begin{aligned} E\delta x + 0\delta y - E\delta z &= E\alpha\,(2E - 2G + F - H) \\ 0\delta x + F\delta y - F\delta z &= F\alpha\,(E - G + 2F - 2H) \end{aligned}\right\}.$$

This simple and obvious example sufficiently illustrates the method of proceeding when the proper multipliers could not be otherwise found.

Some Remarks on those Solutions of Linear Differential Equations which depend on Multiple Types. By E. J. Routh, F.R.S.

[*Added July 20th*, 1883.]

When a set of linear differential equations is given to find x, y, &c. in terms of an independent variable t, the solution in general consists of a set of exponentials real or imaginary, the exponents depending on the roots of a certain determinant. If some of these roots be equal, the exponent is sometimes accompanied by powers of t as factors; and sometimes, instead of these powers of t, we have merely a greater indeterminateness in the constants which accompany the exponential. Occasionally both these occur at once.

We now propose to consider under what circumstances this last event can happen. We begin with a statement of the ordinary method of solution as the shortest way of explaining the notation. We first find that each term of the form $t^n e^{mt}$ is accompanied by one constant. We then discover that this solution becomes indeterminate in a certain case. Obtaining another solution, we find that each term of the form just given is now accompanied by two constants. If the solution again become indeterminate, the term reappears with three constants, and so on. Sometimes we find that some of the powers of t which multiply a given exponential have two, while other powers have three, independent constants. Our object is to construct a rule to determine beforehand the number of independent constants which accompany any term of the solution.

The problem to be solved is enunciated in Art. (30), and the result is stated in Art. (37).

24. The equations which occur in Dynamics are, in general, all of the second order; but, as this restriction is not necessary in what follows, we shall suppose the equations to contain differential coefficients of any order.

Let there be n dependent variables represented by x, y, z, &c., and one independent variable represented by t. If the symbol δ represent differentiation with regard to t, the n equations to find x, y, &c. may be written:

$$\left.\begin{aligned} f_{11}(\delta)\,x + f_{12}(\delta)\,y + f_{13}(\delta)\,z + \ldots &= 0 \\ f_{21}(\delta)\,x + f_{22}(\delta)\,y + f_{23}(\delta)\,z + \ldots &= 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

To solve these, we use the analogy which exists between the rules for combining symbols of differentiation and those of common algebra. Omitting for the moment any one equation—say the first—and proceeding to solve the remaining $n-1$ equations by the rules of common algebra, we find the ratios

$$\frac{1}{I_1(\delta)}\,x = \frac{1}{I_2(\delta)}\,y = \frac{1}{I_3(\delta)}\,z = \&c. = V \quad \ldots\ldots\ldots\ldots\ldots (2),$$

where each of the equalities has been put equal to V. Here we have used the letter I to stand for the minors of the determinant

$$\Delta(\delta) = \begin{vmatrix} f_{11}(\delta) f_{12}(\delta) f_{13}(\delta) \ldots \\ f_{21}(\delta) f_{22}(\delta) f_{23}(\delta) \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The suffix of the letter I indicates the constituent of the omitted equation whose minor is required.

Substituting these values of x, y, z, &c. in the equation previously omitted, we obtain

$$\Delta(\delta)\,V = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

This is an equation to determine a single quantity V as a function of t. We may call V the *type of the solution*. Supposing this equation to be solved by the usual rules, the values of x, y, z, &c. are found by equations (2). Thus we have

$$x = I_1(\delta)\,V, \quad y = I_2(\delta)\,V, \text{ \&c.} \ldots\ldots\ldots\ldots\ldots (5).$$

These operators, $I_1(\delta)$, $I_2(\delta)$, &c., are all *integral and rational functions of* δ; so that when V is once known, all the other operations necessary for the complete solution of the equations are reduced to the one operation of continued differentiation.

25. This method of solution is of course well known, the only novelty being the expression of the results by means of integral and rational functions of the symbol δ. In practice, this novelty will be found to introduce a great simplification into the solution. The type V can always be written down by solving equation (4). It is sometimes

very complicated. In such cases it will be found very convenient to be able to deduce the forms of x, y, z, &c. without having to perform any inverse operation.

26. If the roots of the determinantal equation $\Delta(\delta) = 0$ be m_1, m_2, &c., the type V is known to be

$$V = L_1 e^{m_1 t} + L_2 e^{m_2 t} + \ldots,$$

where L_1, L_2, &c., are arbitrary constants. If we substitute any term of this series in the equations (5), we have

$$x = LI_1(m)\, e^{mt}, \quad y = LI_2(m)\, e^{mt}, \text{ &c.}$$

Thus the values of x, y, &c. corresponding to each term in the type are known.

27. *But we see at once that there is a case of failure.* If one of the roots of the equation $\Delta(\delta) = 0$ make all the minors, $I_1(m)$, $I_2(m)$, &c. equal to zero, the solution becomes incomplete. One constant L disappears from the solution. If all the minors of only one row vanished, we could find the values of x, y, z, &c. by choosing as our operators the minors of some other row. But this cannot be done if all the minors of all the rows are zero.

28. We shall now prove that this indeterminate case cannot occur unless the determinantal equation $\Delta(\delta) = 0$ has equal roots. To show this, we differentiate (3); we find

$$\frac{d\Delta(\delta)}{d\delta} = I_{11}\frac{df_{11}}{d\delta} + I_{12}\frac{df_{12}}{d\delta} + \text{&c.} + I_{21}\frac{df_{21}}{d\delta} + \text{&c.},$$

where the letter I stands for the minor of that constituent of the determinant $\Delta(\delta)$ which is indicated by the suffix. We notice that the right-hand side of this equation vanishes when all the first minors are zero. Thus the equation $\Delta(\delta) = 0$ must have at least two equal roots. In the same way, if the second minors are all zero also, any first minor has two equal roots, and therefore the original equation has three equal roots.

29. We may notice two obvious results. (1) If all the first minors of a determinant have a root α times, the determinant has the root $\alpha+1$ times at least. (2) If a determinant have r equal roots, and all its first, second, &c. minors vanish for these roots, then each of the first minors has the equal root $r-1$ times, each of the second minors $r-2$ times, and so on.

30. We may now consider the following general problem:—

Let the determinant $\Delta(\delta)$ have α roots each equal to m. Let β of these roots make every first minor of $\Delta(\delta)$ equal to zero. Let γ of these last make every second minor equal to zero, and so on. It is required to state

the general form of the solution, and to explain how the α constants in the solution are to be found.

31. First consider the α roots of the determinant $\Delta(\delta)$ each equal to m. By equation (4) of Art. (24) the type of the solution is

$$V = (L_0 + L_1 t \ldots + L_{\alpha-1} t^{\alpha-1}) e^{mt},$$

where the L's are α arbitrary constants. This may be put into the form

$$V = \left(L_0 + L_1 \frac{d}{dm} + \ldots + L_{\alpha-1} \frac{d^{\alpha-1}}{dm^{\alpha-1}}\right) e^{mt}.$$

To find the corresponding values of x, y, &c., we substitute in

$$x = I_1(\delta)\, V, \quad y = I_2(\delta)\, V \ldots,$$

where I_1, I_2, &c. are the minors of the constituents in the first row of the determinant $\Delta(\delta)$. Since the operations $\frac{d}{dm}$ and δ are independent, we may write the result of this substitution in the form

$$x = L_0 [I_1(m) e^{mt}] + L_1 \frac{d}{dm} [I_1(m) e^{mt}] + \ldots + L_{\alpha-1} \frac{d^{\alpha-1}}{dm^{\alpha-1}} [I_1(m) e^{mt}],$$

$$y = L_0 [I_2(m) e^{mt}] + L_1 \frac{d}{dm} [I_2(m) e^{mt}] + \ldots + L_{\alpha-1} \frac{d^{\alpha-1}}{dm^{\alpha-1}} [I_2(m) e^{mt}].$$

&c. = &c.

32. If these first minors are finite, these formulæ contain powers of t from t^0 to $t^{\alpha-1}$, and thus supply the α constants which belong to the α equal roots. If the first minors have β roots equal to m, $I_1(m)$, $I_2(m)$, &c., and their differential coefficients up to the $(\beta - 1)$th are all zero. In this case the powers of t extend only to $t^{\alpha-\beta-1}$, and thus these formulæ do not supply the full number of constants.

When all the first minors have the root α times and all the second minors have the root β times, we know by Art. (29) that $\alpha - \beta - 1$ cannot be negative.

33. To find the proper forms of x, y, &c. when the first minors are all zero, we return to the analogy between operations and quantities, alluded to in Art. (24). We now reject any two equations—say the first two. Solving the remaining $n-2$ equations to find any variable, as z, in terms of x and y, we have

$$I_{23}(\delta)\, x - I_{13}(\delta)\, y + I_{12}(\delta)\, z = 0,$$

with similar equations connecting (x, y, u) (x, y, v), &c. Here the functional symbols are the second minors in the determinant $\Delta(\delta)$ of the constituents of the two omitted equations. Thus I_{rs} is the minor of the rth constituent of the first omitted equation and the sth constituent of the second.

Let now ξ and η be two functions of t which we may call the *double*

type of the solution. Then these equations may all be satisfied by

$$\left.\begin{aligned} x &= I_{12}(\delta)\,\xi \\ y &= I_{12}(\delta)\,\eta \\ z &= I_{13}(\delta)\,\eta - I_{23}(\delta)\,\xi \\ u &= I_{14}(\delta)\,\eta - I_{24}(\delta)\,\xi \\ \&c. &= \&c. \end{aligned}\right\}.$$

We have yet to consider the two equations rejected in this solution. But, substituting these values of x, y, &c., in these equations, we see that the coefficients of ξ and η are the minors of the first and second constituents of these equations. If these minors are not all zero, we have two equations to find ξ and η. These may be solved, as already explained, by expressing them in terms of a single type V. But if all four minors contain the root β times, the equations are satisfied, provided we take

$$\left.\begin{aligned} \xi &= (G_0 + G_1 t + \ldots + G_{\beta-1} t^{\beta-1})\, e^{mt} \\ \eta &= (H_0 + H_1 t + \ldots + H_{\beta-1} t^{\beta-1})\, e^{mt} \end{aligned}\right\}.$$

where the G's and H's constitute 2β arbitrary constants. In addition to these, ξ and η may have other terms depending on a single type of constants. We neglect these as having been already found when discussing the terms arising from the single type V.

The corresponding values of x, y, &c. are found by substitution, and may be written in the form

$$x = G_0 [I_{12}(m)\, e^{mt}] + G_1 \frac{d}{dm} [I_{12}(m) e^{mt}] + \ldots\ldots$$

$$+ G_{\beta-1} \frac{d^{\beta-1}}{dm^{\beta-1}} [I_{12}(m) e^{mt}],$$

with similar expressions for y, &c.

The peculiarity of the solutions which are derived from the double type ξ, η is that the corresponding terms in the expressions for x and y have *independent constants.*

If the second minors which form the operators are all finite, these formulæ contain powers of t up to $t^{\beta-1}$ and supply 2β constants. But if these second minors contain γ roots equal to m, the powers of t extend only to $t^{\beta-\gamma-1}$, and thus the full number of constants has not been found.

34. Thirdly, we have to find the solution when the second minors are zero as well as the first minors. In this case the solution just found becomes insufficient. To find the proper forms of x, y, z, &c., we now reject any three of the differential equations, say the first three, and proceed as before. We find that x, y, z may be expressed by means

of operators which are third minors of the determinant $\Delta(\delta)$. The subjects of operation are *three* independent functions ξ, η, ζ, which we call the *triple type of the solution.* These are such that, if $I(\delta)$ be any of the second minors of the three first constituents of the three omitted equations,

$$I(\delta)\,\xi = 0, \quad I(\delta)\,\eta = 0, \quad I(\delta)\,\zeta = 0.$$

These by hypothesis contain the root γ times. Hence each of the three, ξ, η, ζ, is a series of the form

$$(K_0 + K_1 t + \ldots + K_{\gamma-1} t^{\gamma-1})\, e^{mt},$$

but with independent constants.

35. Each of the sets of values of x, y, &c. given in Arts. (31), (33), and (34) is, of course, *a solution.* The complete solution is really the sum of these partial solutions, provided it has the proper number of constants. We appear, however, to have too many constants. We must therefore examine these, and determine what terms are absolutely zero and what terms are repeated in the several partial solutions.

36. We begin with the solution derived from the type V, Art. (31), by the help of the first minors. Since the first minors have β roots each equal to m, the first β terms of each of the expressions for x, y, &c. are easily seen to be zero. Consider the solution derived from any term L_k, where k lies between $\beta-1$ and 2β. In the case of the variables x and y they are expressions of the form

$$(A_0 + A_1 t + \ldots + A_{k-\beta} t^{k-\beta})\, e^{mt}.$$

All these are evidently included amongst the terms derived from ξ, η by the help of the second minors. The corresponding terms in z, u, &c. must be related to the terms in x, y by the formula given in Art. (33), and are therefore also included in the series derived from ξ, η. Lastly, consider the solution derived from the terms from $L_{2\beta}$ to $L_{\alpha-1}$. They include powers of t from t^{β} to $t^{\alpha-1-\beta}$. These $\alpha-2\beta$ terms are not included in the terms derived from ξ and η, and they supply $\alpha-2\beta$ arbitrary constants.

Secondly, we turn our attention to the solution derived from the double type ξ, η by the help of the second minors (Art. 33). Each of these second minors has γ roots each equal to m; hence, by the same reasoning as before, the first γ terms of the series for x and y are zero, and the highest power of t is $\beta-1-\gamma$ instead of $\beta-1$. In consequence of this, the terms of the series derived from the single type V, and not included in those derived from the double type ξ, η, now extend their powers of t from $t^{\beta-\gamma}$ to $t^{\alpha-1-\beta}$. There are therefore $\alpha-2\beta+\gamma$ such terms instead of $\alpha-2\beta$.

37. The same reasoning applies to all the other partial solutions derived from the triple and higher types. *We therefore conclude that*

the partial solution derived from a single type by operating with the first minors of the first row of the fundamental determinant supplies $\alpha-2\beta+\gamma$ *terms not included in the solutions which follow. These supply as many arbitrary constants. The partial solution derived from a double type by operating with the second minors of the two first rows of the fundamental determinant supplies* $\beta-2\gamma+\delta$ *terms not included in the solutions which follow. These supply twice as many constants. The partial solution derived from a triple type by operating with the third minors of the three first rows supplies* $\gamma-2\delta+\epsilon$ *terms and thrice as many constants, and so on.*

Thus suppose (for example) the fourth minors are not all zero; the number of constants supplied by each of the several partial solutions is indicated by the terms of the series

$$(\alpha-2\beta+\gamma)+2(\beta-2\gamma+\delta)+3(\gamma-2\delta)+4\delta.$$

If none of the terms of this series are negative, we have obtained a series of partial solutions containing the proper number of constants. This point we now proceed to discuss.

38. If a determinant contain the root just α times, if the first minors of the two first constituents of the two first rows contain the root just β times, if the second minor of these four constituents contain the root just γ times, then $\alpha-2\beta+\gamma$ is positive.

To prove this, let Δ be the determinant, I_1, I_2, J_1, J_2 the four first minors, Δ_2 the second minor. Then we know that $\Delta\Delta_2 = I_1 J_2 - I_2 J_1$. The left-hand side contains the root just $\alpha+\gamma$ times, the right-hand side contains the root at least 2β times. Hence $\alpha+\gamma-2\beta$ is positive.

In the same way we may show, on similar suppositions, that $\beta-2\gamma+\delta$ is positive, and so on.

39. The partial solutions thus obtained may be exhibited thus:—

Terms derived from a single type:

$$x = (A_0 + A_1 t + \ldots + A t^{\alpha-\beta-1})\, e^{mt}.$$

There are similar expressions for y, z, &c. with multiples of these A's for coefficients.

Terms derived from a double type:

$$x = (B_0 + B_1 t + \ldots + B t^{\beta-\gamma-1}) e^{mt},$$
$$y = (C_0 + C_1 t + \ldots + C t^{\beta-\gamma-1}) e^{mt}.$$

There are similar expressions for z, u, &c. with linear functions of these B's and C's for coefficients.

Terms derived from a triple type:

$$x = (D_0 + D_1 t + \ldots + D t^{\gamma-\delta-1})\, e^{mt},$$
$$y = (E_0 + E_1 t + \ldots + E t^{\gamma-\delta-1})\, e^{mt},$$
$$z = (F_0 + F_1 t + \ldots + F t^{\gamma-\delta-1}) e^{mt}.$$

There are similar expressions for the other coordinates with linear functions of these D's, E's, and F's as coefficients.

Terms derived from a quadruple type:

$$x = (G_0 + G_1 t + \ldots + G t^{\delta-1}) e^{mt},$$
$$y = (H_0 + H_1 t + \ldots + H t^{\delta-1}) e^{mt},$$
$$z = (K_0 + L_1 t + \ldots + K t^{\delta-1}) e^{mt},$$
$$u = (L_0 + L_1 t + \ldots + L t^{\delta-1}) e^{mt}.$$

There are similar expressions for the other coordinates whose coefficients are linear functions of these arbitrary constants.

40. Conversely, suppose it is given that we have such a solution as that just found. We have here terms derived from the quadruple type containing $(\delta-1)$ powers of t accompanied by independent constants in any four coordinates. Putting any three of these coordinates equal to zero, the differential equations are satisfied by a solution depending on a single type. Thus we have n equations containing $n-3$ coordinates, all satisfied by values of the coordinates which contain powers of t up to the $(\delta-1)$th. This proves that all the third minors which can be formed from these equations must be zero, and each of these minors must contain the root δ times.

From this we infer by Art. (29) that every second minor must contain the root $\delta+1$ times. But besides these we have terms derived from the triple type which contain powers of t up to the $(\gamma-\delta-1)$th. Reasoning in the same way as before, we see that every second minor must have the root $\gamma-\delta-1$ times. These are in addition to the $\delta+1$ times already found. Hence every second minor has the root γ times.

That these are in addition to the $\delta+1$ roots previously found is clear, because we may treat the terms derived from the different types as the limiting cases of unequal roots. But we may also deduce the same thing from Art. (32). It is there shown, that when any determinant (such as this second minor) has α roots and its minors have the root β times, the powers of t which result rise only to $\alpha-\beta-1$.

In the same way every first minor has the root β times, and so on.

41. We may notice as a corollary of this theory that the solution cannot contain terms in which the high powers of t depend on a larger type than the low powers of t. For example, if the term $t^n e^{mt}$ occur accompanied by k independent constants, then all the lower powers of t which multiply the same exponential must be accompanied by at least k independent constants.

42. In some dynamical problems it is well known that, though the fundamental determinant has α equal roots, yet there are no terms in the solution with powers of t. We may now determine the conditions that this may occur.

We see by Art. (39) that, unless every first minor has the root $\alpha-1$ times at least, a solution can be deduced from the first minors which has some power of t greater than zero in the coefficient. Again, unless every second minor has the root $\alpha-2$ times at least, a solution can be deduced from the second minors with some power of t in the coefficient. On the whole, we infer that when α equal roots occur in the determinant, and the terms in the solution with t as a factor are to be absent, it is necessary as well as sufficient that all the first, second, &c. minors up to the $(\alpha-1)$th should be zero.

---

*Investigation of the Character of the Equilibrium of an Incompressible Heavy Fluid of Variable Density.** By Lord Rayleigh.

[*Read April 12th*, 1883.]

The well-known condition of equilibrium requires that the fluid be arranged in horizontal strata, so that its density σ is a function of the vertical coordinate z only. If this state of things be slightly departed from, we may regard the actual density at any point x, y, z as equal to $\sigma+\rho$, where ρ is a function of x, y, z, and the time t, which always remains small during the period contemplated. The component velocities u, v, w are equally to be regarded as small; they are connected by the equation of continuity

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad \text{.....(1).}$$

The equilibrium pressure p is a function of z only. If the actual pressure be called $p+\delta p$, the dynamical equations become, with omission of the squares of small quantities,

$$\frac{d\delta p}{dx}=-\sigma\frac{du}{dt}, \quad \frac{d\delta p}{dy}=-\sigma\frac{dv}{dt}, \quad \frac{d\delta p}{dz}=-g\rho-\sigma\frac{dw}{dt} \quad \text{.....(2).}$$

One further equation is supplied by the condition that the density of every particle remains unchanged.

Thus

$$\frac{d\rho}{dt}+w\frac{d\sigma}{dz}=0 \quad \text{.....(3).}$$

---

* These calculations were written out in 1880, in order to illustrate the theory of cirrous clouds propounded by the late Prof. Jevons (*Phil. Mag.*, xiv., p. 22, 1857). Pressure of other work has prevented me hitherto from pursuing the subject.

By Fourier's theorem, and the general theory of disturbed equilibrium, we know that the complete solution of the present problem may be decomposed into partial solutions, for any one of which the variable quantities considered as functions of x vary as $e^{i\kappa x}$, as functions of y vary as $e^{i\kappa' y}$, and as functions of t vary as e^{int}. The *wave-lengths* of the disturbances parallel to x and y are λ, λ'; where $\lambda = 2\pi/\kappa$, $\lambda' = 2\pi/\kappa'$.

The introduction of these suppositions into (1), (2), and (3) leads to

$$i\kappa u + i\kappa' v + \frac{dw}{dz} = 0 \quad \text{.....(4),}$$

$$\kappa \delta p = -n\sigma u, \quad \kappa' \delta p = -n\sigma v, \quad \frac{d\delta p}{dz} = -g\rho - in\sigma w \quad \text{.....(5),}$$

$$in\rho + w\frac{d\sigma}{dz} = 0 \quad \text{.....(6).}$$

Eliminating u and v between (4) and the two first of equations (5), we get

$$i(\kappa^2 + \kappa'^2)\,\delta p - n\sigma \frac{dw}{dz} = 0 \quad \text{.....(7).}$$

Next eliminating δp between (7) and the last of equations (5), we find

$$i(\kappa^2 + \kappa'^2)(g\rho + in\sigma w) + n\frac{d}{dz}\left(\sigma \frac{dw}{dz}\right) = 0 \quad \text{.....(8).}$$

Finally between (6) and (8) we eliminate ρ, and thus obtain

$$\frac{d}{dz}\left(\sigma \frac{dw}{dz}\right) - (\kappa^2 + \kappa'^2)\left(\frac{g}{n^2}\frac{d\sigma}{dz} + \sigma\right) w = 0 \quad \text{.....(9);}$$

or, as it may be also written,

$$\frac{d^2w}{dz^2} - (\kappa^2 + \kappa'^2)\,w + \frac{d\sigma}{\sigma dz}\left\{\frac{dw}{dz} - (\kappa^2 + \kappa'^2)\frac{g}{n^2}w\right\} = 0 \quad \text{.....(10).}$$

We will first apply this equation to the well-known case of two uniform fluids of densities σ_1, σ_2, separated by a horizontal boundary ($z = 0$), and for brevity we will omit to write κ'. For both regions of fluid, the general equation (10) reduces to

$$\frac{d^2w}{dz^2} - \kappa^2 w = 0 \quad \text{.....(11),}$$

of which the solution is

$$w = Ae^{\kappa z} + Be^{-\kappa z} \quad \text{.....(12).}$$

By the condition at infinity, we are to take for the upper fluid $A = 0$, and for the lower $B = 0$. Moreover by continuity the value of w must be the same for both fluids at the separating surface. Thus we may write for the upper fluid $w = Be^{-\kappa z}$, and for the lower $w = Be^{\kappa z}$. The second boundary condition is obtained by integrating equation

(9) across the surface of transition. Thus

$$\left[\sigma \frac{dw}{dz}\right]_2 - \left[\sigma \frac{dw}{dz}\right]_1 - \frac{g\kappa^2}{n^2}(\sigma_2 - \sigma_1) = 0;$$

whence
$$n^2 = g\kappa \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$
the known solution.

If the upper fluid be the lighter, $\sigma_2 < \sigma_1$, and n^2 is positive. This indicates stability with harmonic oscillations, whose frequency increases without limit with κ; that is, as the wave-length diminishes. If, on the other hand, $\sigma_2 > \sigma_1$, the equilibrium is unstable, and the instability (measured by the rate at which a small disturbance is multiplied in a given time) is greater the smaller the wave-length. If the disturbance be not limited to two dimensions, we have simply to replace κ by $\sqrt{(\kappa^2 + \kappa'^2)}$.

We know from the general theory that only real values of n^2 are admissible in (9), and that if $\frac{d\sigma}{dz}$ be negative throughout, all the values of n^2 are positive, but if $\frac{d\sigma}{dz}$ be positive throughout, all the values of n^2 are negative. In order to prove this from the equation, suppose that w and w' are two solutions corresponding to different values of n^2, say n^2 and n'^2. Then

$$\int w' \frac{d}{dz}\left(\sigma \frac{dw}{dz}\right) dz = \kappa^2 \int \left(\frac{g}{n^2}\frac{d\sigma}{dy} + \sigma\right) ww'\, dz,$$

or, on integration by parts between two finite or infinite limits for which w, w' vanish,

$$\int \sigma \frac{dw}{dz}\frac{dw'}{dz} dz + \kappa^2 \int \sigma w w'\, dz + \kappa^2 \frac{g}{n^2} \int \frac{d\sigma}{dz} ww'\, dz = 0 \quad \ldots\ldots (14).$$

In this equation w and w' may be interchanged if n'^2 be written for n^2. Hence
$$\int \sigma \frac{dw}{dz}\frac{dw'}{dz} dz + \kappa^2 \int \sigma w w'\, dz = 0 \ldots\ldots\ldots\ldots\ldots\ldots (15),$$
$$\int \frac{d\sigma}{dz} ww'\, dz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

If now n^2 could be complex, there would be two solutions of the form $w = \alpha + i\beta$, $w' = \alpha - i\beta$, and equation (15) would become

$$\int \sigma \left\{ \left(\frac{d\alpha}{dz}\right)^2 + \left(\frac{d\beta}{dz}\right)^2 \right\} dz + \kappa^2 \int \sigma\, (\alpha^2 + \beta^2)\, dz = 0,$$

which cannot be true if, as we suppose, σ is everywhere positive.

Again, suppose in (14) that $w' = w$. Thus

$$\int \sigma \left\{ \left(\frac{dw}{dz}\right)^2 + \kappa^2 w^2 \right\} dz + \frac{\kappa^2 g}{n^2} \int \frac{d\sigma}{dz} w^2 dz = 0 \ldots\ldots\ldots\ldots\ldots (17),$$

from which it is evident that, if $\frac{d\sigma}{dz}$ be of one sign throughout, n^2 can only be of the opposite sign.

These conclusions are limited to the cases for which every mode of disturbance is stable, or every mode unstable; but we know that if $\frac{d\sigma}{dz}$ be *anywhere* positive, instability must ensue. To see this from equation (9), we may regard it as the condition (according to the methods of the Calculus of Variations) that $\int \frac{d\sigma}{dz} w^2 dz$ is a maximum or minimum, while $\int \sigma \left\{ \left(\frac{dw}{dz}\right)^2 + \kappa^2 w^2 \right\} dz$ is given, $-\frac{n^2}{g\kappa^2}$ being the then value of the ratio of the integrals. If $d\sigma/dz$ be anywhere positive, the first integral admits of a positive value, and therefore of a positive maximum, so that one value at least of n^2 is negative, and one mode of disturbance is unstable.

The simplest case of a variable density which we can consider is that obtained by supposing $\frac{d\sigma}{\sigma dz}$ to be constant, equal say to β, or, on integration,

$$\sigma = \sigma_0 e^{\beta z} \quad \ldots\ldots (18);$$

so that all strata of equal thickness are similarly constituted, differing only in absolute density. In this case, with omission of κ' as before (10) becomes

$$\frac{d^2w}{dz^2} + \beta \frac{dw}{dz} - \kappa^2 \left(1 + \frac{g\beta}{n^2}\right) w = 0 \quad \ldots\ldots (19).$$

If m_1, m_2 be the roots of

$$m^2 + \beta m - \kappa^2 (1 + g\beta n^{-2}) = 0 \quad \ldots\ldots (20),$$

the general solution of (19) is

$$w = Ae^{m_1 z} + Be^{m_2 z} \quad \ldots\ldots (21),$$

A and B being arbitrary constants.

Let us now suppose that the fluid is bounded by impenetrable horizontal planes at $z = 0$ and at $z = l$. Since w vanishes with z, $B = -A$, so that (21) becomes

$$w = A (e^{m_1 z} - e^{m_2 z}) \quad \ldots\ldots (22).$$

Again, since w vanishes when $z = l$, $e^{m_1 l} - e^{m_2 l} = 0$, or $e^{(m_1 - m_2) l} = 1$, whence

$$(m_1 - m_2)\, l = 2a i \pi \quad \ldots\ldots (23),$$

a being an integer. Thus (22) may be written

$$w = Ae^{\frac{1}{2}(m_1 + m_2) z} \left\{ e^{\frac{1}{2}(m_1 - m_2) z} - e^{-\frac{1}{2}(m_1 - m_2) z} \right\} = A' e^{-\frac{1}{2}\beta z} \sin a \frac{\pi z}{l} \quad \ldots\ldots (24),$$

by (20), (23), A' being a new arbitrary constant. The values of n corresponding to the various values of α are obtained by comparison of (20) and (23). From the former

$$(m_1-m_2)^2 l^2 = \beta^2 l^2 + 4\kappa^2 l^2 (1+g\beta n^{-2}) \quad \ldots\ldots(25),$$

so that

$$\kappa^2 l^2 (1+g\beta n^{-2}) = -\tfrac{1}{4}\beta^2 l^2 - \alpha^2\pi^2 \quad \ldots\ldots(26),$$

or

$$\frac{g\beta}{n^2} = -\frac{\tfrac{1}{4}\beta^2 l^2 + \kappa^2 l^2 + \alpha^2\pi^2}{\kappa^2 l^2} \quad \ldots\ldots(27).$$

From (27) we see that the disturbances are all stable if β is negative, that is, if the density diminishes upwards, and that in the contrary case they are all unstable. The smallest admissible value of α is unity, and this corresponds to the greatest numerical value of n^2. Contrary to what is met with in most vibrating systems, there is (in the case of stability) a limit on the side of rapidity of vibration, but none on the side of slowness. In the case of instability we are principally interested in the mode for which the instability is greatest, and this also corresponds to the unit value of α. When α is greater than unity, there are internal nodal planes, as appears from (24).

If l, κ, and α are given, n^2 is numerically greatest when β is such that

$$\tfrac{1}{4}\beta^2 l^2 = \kappa^2 l^2 + \alpha^2\pi^2.$$

If l, α, β be regarded as given, n^2 increases numerically from zero when κ is zero, up to a finite limit when κ is infinite; or, in the case of stability, as the wave-length diminishes from ∞ to 0, the frequency of vibration rises from 0 to a finite value, given by

$$n^2 = -g\beta \quad \ldots\ldots(28),$$

which is independent both of α and of l. These vibrations are isochronous with the vibrations of a pendulum whose length is equal to the distance between two strata, of which the densities are as $e:1$.

If the disturbance be not limited to two dimensions, we must write $\sqrt{(\kappa^2+\kappa'^2)}$ for κ^2. The completely expressed value of w, corresponding to one normal mode of disturbance, is then

$$w = Ae^{-\frac{1}{2}\beta z} \sin\frac{\alpha\pi z}{l} \cos\frac{2\pi(x-x_0)}{\lambda} \cos\frac{2\pi(y-y_0)}{\lambda'} \cos n(t-t_0) \ldots(29).$$

We will now apply the solution to the investigation of the case in which the density for all values of z less than 0 is σ_1, and for all values of z greater than l is σ_2, the transition from the one density to the other being in accordance with the law $\sigma = \sigma_1 e^{\beta z}$, so that

$$\sigma_2 = \sigma_1 e^{\beta l} \quad \ldots\ldots(30).$$

When $z>l$, $w \propto e^{-\kappa z}$, so that for $z=l$, $dw/w\,dz = -\kappa$; similarly for the lower fluid, when $z=0$, $dw/w\,dz = +\kappa$. Thus, by (21), the boundary

conditions are $m_1 A e^{m_1 l} + m_2 B e^{m_2 l} = -\kappa (A e^{m_1 l} + B e^{m_2 l})$,

$$m_1 A + m_2 B = +\kappa (A + B),$$

whence, by elimination of $A : B$,

$$e^{m_1 l} (m_1 + \kappa)(m_2 - \kappa) = e^{m_2 l} (m_2 + \kappa)(m_1 - \kappa).$$

This, in connection with (20), determines the admissible values of n. It may be written

$$\frac{m_1 m_2 + \kappa (m_2 - m_1) - \kappa^2}{m_1 m_2 - \kappa (m_2 - m_1) - \kappa^2} = e^{(m_2 - m_1) l},$$

or

$$\frac{\kappa (m_2 - m_1)}{m_1 m_2 - \kappa^2} = \tanh \tfrac{1}{2} (m_2 - m_1)\, l.$$

By (20) this may be put into the form

$$\frac{\kappa (m_2 - m_1)}{\frac{1}{4}\beta^2 - \kappa^2 - \frac{1}{4}(m_2 - m_1)^2} = \tanh \tfrac{1}{2} (m_2 - m_1)\, l,$$

or, if for brevity we write θ for $(m_2 - m_1)\, l$,

$$\frac{\kappa l \,.\, \theta}{\frac{1}{4}\beta^2 l^2 - \kappa^2 l^2 - \frac{1}{4}\theta^2} = \tanh \tfrac{1}{2}\theta \qquad \text{.....(31).}$$

This equation determines θ; and then, by (20),

$$\theta^2 = l^2 (m_2 - m_1)^2 = l^2 \{(m_2 + m_1)^2 - 4 m_1 m_2\}$$
$$= \beta^2 l^2 + 4\kappa^2 l^2 (1 + g\beta n^{-2}) \qquad \text{.....(32),}$$

giving n in terms of θ.

Before going farther, we may verify these results by applying them to the case of a sudden transition, for which l vanishes, while βl remains finite. The principal solution of (31) gives $\theta^2 = \beta^2 l^2$ approximately, so that

$$\tfrac{1}{4}\beta^2 l^2 - \tfrac{1}{4}\theta^2 = \kappa l \,.\, \beta l \,.\, \coth \tfrac{1}{2}\beta l.$$

Using this in (32), we get

$$g\beta n^{-2} = -\kappa^{-1}\beta \coth \tfrac{1}{2}\beta l,$$

whence

$$n^2 = -g\kappa \tanh \tfrac{1}{2}\beta l = -g\kappa \frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1},$$

as before.

Other solutions of (31) are obtained by supposing $\theta^{-1} \tanh \frac{1}{2}\theta$ to vanish, whence $\theta = i \,.\, \alpha \,.\, 2\pi$, α being an integer other than zero, These are of no importance, as the corresponding values of n vanish.

When the layer of transition is of finite thickness, the general solution expressed by (31), (32) is rather complicated. A simplification, which does not involve much loss of interest, may be effected by supposing that the whole change of density is small, so that (31), (32)

become $$\frac{\kappa l \,.\, \theta}{\kappa^2 l^2 + \frac{1}{4}\theta^2} = -\tanh \tfrac{1}{2}\theta \quad \text{.....(33),}$$

$$g\beta n^{-2} = \frac{\theta^2}{4\kappa^2 l^2} - 1 \quad \text{.....(34).}$$

From (33), $$\pm \frac{\kappa l - \frac{1}{2}\theta}{\kappa l + \frac{1}{2}\theta} = e^{i\theta},$$

whence $$-\tanh \tfrac{1}{4}\theta = \frac{\theta}{2\kappa l}, \text{ or } \frac{2\kappa l}{\theta} \quad \text{.....(35).}$$

Equation (35) cannot be satisfied by any real value of θ. If we write $\theta = i\phi$, we get in place of it,

$$\tan \tfrac{1}{4}\phi = -\frac{\phi}{2\kappa l}, \text{ or } \frac{2\kappa l}{\phi} \quad \text{.....(36);}$$

and in place of (34), $$g\beta n^{-2} = -\frac{\phi^2}{4\kappa^2 l^2} - 1 \quad \text{.....(37).}$$

The series of admissible values of ϕ, given by (36), extends to infinity, but the higher roots correspond to small values of n^2, which are of little interest. Whether the equilibrium be stable or unstable, the most important root is the smallest. It lies in the first quadrant, and is given by the second alternative of (36). The progress of n^2 as a function of κl is easily traced. When κl is small, $\phi^2 = 8\kappa l$, and $g\beta n^{-2} = -2\,(\kappa l)^{-1}$, which leads to $n^2 = -g\kappa\,\frac{\sigma_2 - \sigma_1}{2\sigma}$, the known result for a rapid transition. As κl increases, $\frac{1}{4}\phi$ ranges from 0 to $\frac{1}{2}\pi$, and $\frac{\phi^2}{4\kappa^2 l^2}$ or $\cot^2 \frac{1}{4}\phi$ ranges from infinity to zero. Thus the numerical value of n^2 continually increases, until for an infinitely small wave-length it approaches the finite limit $-g\beta$, beyond which it cannot pass. The principal result of the substitution of a gradual for an abrupt transition is to arrest the further increase of n^2, after the wave-length has diminished so far as to become comparable in magnitude with the thickness of the layer of transition. In the case of the limiting value of n^2, the length of the equivalent pendulum is

$$l \div (\log \sigma_2 - \log \sigma_1).$$

If, for example, the extreme difference of densities amounted to one per cent., the length of the equivalent pendulum would be 100 times the thickness of the layer of transition.

For actual calculation (36), (37) may advantageously be written

$$\tfrac{1}{2}\kappa l = \tfrac{1}{4}\phi \times \tan \tfrac{1}{4}\phi \quad \text{.....(38),}$$

$$-\tfrac{1}{2}\kappa g \,.\, \beta l \,.\, n^{-2} = \tfrac{1}{2}\kappa l \div \sin^2 \tfrac{1}{4}\phi \quad \text{.....(39),}$$

the right-hand member of (39) being equal to unity, when κl is small.

Ascribing arbitrary values to $\frac{1}{4}\phi$, we can readily calculate corresponding values of κl and $\frac{1}{2}\kappa l / \sin^2 \frac{1}{4}\phi$, and thus exhibit the effect upon the equilibrium of a gradual increase in the thickness of the layer of transition, the extreme densities (determined by βl) and the wave-length being given.

$\frac{1}{4}\phi$	κl	$\frac{1}{2}\kappa l / \sin^2 \frac{1}{4}\phi$
0	κl	1·000
10°	·06155	1·021
20°	·2541	1·086
30°	·6046	1·209
40°	1·172	1·418
50°	2·080	1·772
60°	3·628	2·418
70°	6·713	3·801
80°	15·838	8·165
90°	κl	$\frac{1}{2}\kappa l$

Equations of the Loci of the Intersections of three Tangent Lines and of three Tangent Planes to any Quadric $u = 0$. *By* Professor Wolstenholme.

[*Read April 12th*, 1883.]

If $u = 0$ be the rational equation of the second degree of any quadric, the equation of the tangent cone whose vertex is (xyz) is (using XYZ for current coordinates), when referred to its vertex as origin,

$$4uU = \left(X\frac{du}{dx} + Y\frac{du}{dy} + Z\frac{du}{dz}\right)^2;$$

and the coefficient of X^2 in this is

$$2u\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2;$$

that of $2YZ$ is

$$2u\frac{d^2u}{dy\,dz} - \frac{du}{dy}\frac{du}{dz},$$

Hence if we write A, B, C, F, G, H for

$$\frac{d^2u^{\frac{1}{2}}}{dx^2}, \quad \frac{d^2u^{\frac{1}{2}}}{dy^2}, \quad \frac{d^2u^{\frac{1}{2}}}{dz^2}, \quad \frac{d^2u^{\frac{1}{2}}}{dy\,dz}, \quad \frac{d^2u^{\frac{1}{2}}}{dz\,dx}, \quad \frac{d^2u^{\frac{1}{2}}}{dx\,dy},$$

the equation may be written

$$AX^2+BY^2+CZ^2+2FYZ+2GZX+2HXY=0.$$

If now the axes be inclined at angles α, β, γ, and we form the discriminant of

$$\lambda\,(X^2+Y^2+Z^2+2YZ\cos\alpha+2ZX\cos\beta+2XY\cos\gamma)\\ +AX^2\ldots+2FYZ\ldots,$$

we see that the coefficient of λ^2 is

$$A\sin^2\alpha+B\sin^2\beta+C\sin^2\gamma-2F\,(\cos\alpha-\cos\beta\cos\gamma)\\ -2G\,(\cos\beta-\cos\gamma\cos\alpha)-2H\,(\cos\gamma-\cos\alpha\cos\beta),$$

and that of λ is

$$BC-F^2+CA-G^2+AB-H^2-2\,(AF-GH)\cos\alpha-2\,(BG-HF)\cos\beta\\ -2\,(CH-FG)\cos\gamma.$$

But, when the coefficient of $\lambda^2=0$, three generators mutually at right angles can be drawn on the cone; and when the coefficient of $\lambda=0$, three tangent planes mutually at right angles can be drawn touching the cone. Hence the locus of the point of concurrence of three tangent lines, two and two at right angles to any quadric $u=0$, is

$$A\sin^2\alpha+B\sin^2\beta+C\sin^2\gamma=2F\,(\cos\alpha-\cos\beta\cos\gamma)\\ +2G\,(\cos\beta-\cos\gamma\cos\alpha)+2H\,(\cos\gamma-\cos\alpha\cos\beta),$$

and the locus of the point of intersection of three planes, two and two at right angles, is

$$BC-F^2+CA-G^2+AB-H^2\\ =2\,(AF-GH)\cos\alpha+2\,(BG-HF)\cos\beta+2\,(CH-FG)\cos\gamma;$$

where $$A=\frac{d^2u^{\frac{1}{3}}}{dx^2},\ \&c.,\quad F=\frac{d^2u^{\frac{1}{3}}}{dy\,dz},\ \&c.$$

For rectangular axes the equations are therefore

$$\frac{d^2u^{\frac{1}{3}}}{dx^2}+\frac{d^2u^{\frac{1}{3}}}{dy^2}+\frac{d^2u^{\frac{1}{3}}}{dz^2}=0\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{d^2u^{\frac{1}{3}}}{dy^2}\frac{d^2u^{\frac{1}{3}}}{dz^2}-\left(\frac{d^2u^{\frac{1}{3}}}{dy\,dz}\right)^2+\frac{d^2u^{\frac{1}{3}}}{dz^2}\frac{d^2u^{\frac{1}{3}}}{dx^2}-\left(\frac{d^2u^{\frac{1}{3}}}{dz\,dx}\right)^2+\frac{d^2u^{\frac{1}{3}}}{dx^2}\frac{d^2u^{\frac{1}{3}}}{dy^2}-\left(\frac{d^2u^{\frac{1}{3}}}{dx\,dy}\right)^2=0\\ \ldots\ldots\ldots\ldots(2);$$

which are easily tested on the common form

$$u=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1,$$

and will be found to give the correct results.

Calculation of the Equation which determines the Anharmonic Ratios of the Roots of a Quintic. By M. J. M. Hill, M.A.

[*Read April 12th*, 1883.]

Abstract.

The object of this paper is to calculate the equation whose roots are the anharmonic ratios of the roots of a quintic.

I. Let the roots of a quintic be denoted by a_1, a_2, a_3, a_4, a_5. Four roots may be selected from them in five ways. By taking these in different orders, six anharmonic ratios may be formed. Therefore from the five roots there can be formed, in all, thirty anharmonic ratios. The equation to be calculated will therefore be of the thirtieth degree.

The anharmonic ratios, however, are not all independent. It is known that the six anharmonic ratios of four quantities are known when one of their number is known; and it is shown in this paper that the thirty anharmonic ratios of five quantities are known when two (not any two) of their number are known.

The relations connecting them are exceedingly simple.

Those connecting the anharmonic ratios of four roots are well known, and are given farther on in the paper; and it is here shown that, if λ_1 be the anharmonic ratio $\frac{(a_3-a_4)(a_2-a_5)}{(a_2-a_3)(a_5-a_4)}$ of the roots a_2, a_3, a_4, a_5, and if λ_2, λ_3, λ_4, λ_5 be obtained from this expression by cyclical changes of the suffixes of the roots, then

$$\lambda_1\lambda_2+\lambda_4 = \lambda_2\lambda_3+\lambda_5 = \lambda_3\lambda_4+\lambda_1 = \lambda_4\lambda_5+\lambda_2 = \lambda_5\lambda_1+\lambda_3 = 1.$$

These relations may be conveniently remembered thus:—Let the anharmonic ratios λ_1, λ_2, λ_3, λ_4, λ_5 be written down at the vertices of a regular pentagon in order; then the product of the two which stand at the extremities of any side of the pentagon, added to the one which stands at the opposite vertex, will always be equal to unity.

These relations are sufficient to determine any three of the quantities λ_1, λ_2, λ_3, λ_4, λ_5 in terms of the remaining two.

But the thirty anharmonic ratios depend on λ_1, λ_2, λ_3, λ_4, λ_5. Therefore they depend on any two of the five quantities λ_1, λ_2, λ_3, λ_4, λ_5.

It might have been expected beforehand that they would depend on two quantities, for an anharmonic ratio is an absolute invariant, and a quintic has only two absolute invariants.

II. (1) The calculation of the equation proceeds in the following way.

First, the equation, whose roots are all the anharmonic ratios depending on λ_1 only, is reduced to the form

$$(\lambda^2-\lambda)^2\left[\Lambda-\frac{(\lambda_1^2-\lambda_1+1)^3}{(\lambda_1^2-\lambda_1)^2}\right]=0,$$

where

$$\Lambda=\frac{(\lambda^2-\lambda+1)^3}{(\lambda^2-\lambda)^2}.$$

Then the equation appears as one of the fifth degree only in Λ.

(2) The expression for one of its roots, viz. $\frac{(\lambda_1^2-\lambda_1+1)^3}{(\lambda-\lambda_1)^2}$, is thrown into the form

$$\frac{[(\overline{23})(\overline{24})(\overline{53})(\overline{54})+(\overline{24})(\overline{25})(\overline{34})(\overline{35})+(\overline{25})(\overline{23})(\overline{45})(\overline{43})]^3}{[(\overline{23})(\overline{24})(\overline{25})(\overline{34})(\overline{35})(\overline{45})]^2},$$

(where for brevity the excess of the r^{th} over the s^{th} root is written $\overline{rs}$), in which it can be seen to be symmetrical with regard to α_2, α_3, α_4, α_5.

[It should be noticed, in passing, although no further use is made of the fact, that from the symmetrical function of the four quantities α_2, α_3, α_4, α_5, there can be obtained by cyclical changes of the suffixes of the roots, four other functions, each symmetrical with regard to four roots; but that all these functions will be determined when two quantities only are known (viz., any two independent anharmonic ratios of the five roots α_1, α_2, α_3, α_4, α_5).]

III. The expression $\frac{(\lambda_1^2-\lambda_1+1)^3}{(\lambda^2-\lambda_1)^2}$, as can be seen from its expression in terms of the roots, is an absolute invariant, because it is not changed by the most general linear transformation. It is also symmetrical with regard to the four quantities α_2, α_3, α_4, α_5.

Now, the roots of the equation in Λ are the above quantity, and four others obtained by cyclical changes of the suffixes of the roots. Therefore, if the equation in Λ be formed, its coefficients will be symmetrical functions of all the five roots, and at the same time absolute invariants of the quintic. Therefore they may be expressed in terms of two absolute invariants of the quintic.

The coefficients can be expressed as fractions, the numerators and denominators of which are invariants. The weight of the numerator will always be equal to that of the denominator, and will be 20 or 40 or 60.

IV. The coefficients of the equation in Λ will be found to be the sums of numerical multiples of powers and products of the absolute invariants $\frac{A}{B}$ and $\frac{C}{B^{\frac{5}{2}}}$, where

$$A=\Sigma\,(\overline{12})^4\,(\overline{34})^2\,(\overline{35})^2\,(\overline{45})^2,$$

$$B = [(\overline{12})(\overline{13})(\overline{14})(\overline{15})(\overline{23})(\overline{24})(\overline{25})(\overline{34})(\overline{35})(\overline{45})]^2,$$
$$C = B^2 \Sigma (\overline{12})^{-4} (\overline{34})^{-2} (\overline{35})^{-2} (\overline{45})^{-2},$$

given in Salmon's *Higher Algebra*, Third Edition, Art. 241.

V. To calculate the values of the coefficients, the quintic is supposed to take the form $x(x^2-a^2)(x^2-b^2)$. It is then shown that there exists no identical relation between the invariants above employed whose weight is 20 or 30 or 40 or 60; and therefore this form may be safely employed in all that follows.

VI. The coefficients of the equation in Λ are calculated in terms of a and b.

VII. The values of the coefficients are determined in terms of the invariants given above, the resulting equation being as follows:—

$$\Lambda^5 - \left(\frac{140}{2^4} + \frac{1}{2^4}\frac{A^2}{B}\right)\Lambda^4 + \left(\frac{480}{2^5} + \frac{24}{2^5}\frac{A^2}{B} + \frac{1}{2}\frac{CA}{B^2}\right)\Lambda^3$$
$$- \left(\frac{2720}{2^8} + \frac{464}{2^8}\frac{A^2}{B} + \frac{2}{2^8}\frac{A^4}{B^2} + \frac{56}{2^4}\frac{CA}{B^2} + \frac{C^2}{B^3}\right)\Lambda^2$$
$$+ \left(\frac{1760}{2^9} - \frac{1456}{2^9}\frac{A^2}{B} + \frac{30}{2^9}\frac{A^4}{B^2} - \frac{36}{2^5}\frac{CA}{B^2} + \frac{1}{2^5}\frac{CA^3}{B^3}\right)\Lambda$$
$$- \left(\frac{12}{2^4} + \frac{1}{2^4}\frac{A^2}{B}\right)^3 = 0.$$

VIII. The relations between the invariants A, B, C, and the invariants J, D, L, employed in Salmon's *Higher Algebra*, are determined, and the above equation, when expressed in terms of them, takes the form

$$\Lambda^5 - \left(\frac{35}{2^2} + \frac{5^3}{2^2}\frac{J^2}{D}\right)\Lambda^4 + \left(15 + \frac{5^4}{2}\frac{J^2}{D} + \frac{5^4}{2}\frac{J^4}{D^2} - \frac{2^{10}5^4 JL}{D^2}\right)\Lambda^3$$
$$- \left(\frac{85}{2^3} + 2^2 5^3\frac{J^2}{D} + 49\frac{5^4}{2^3}\frac{J^4}{D^2} + \frac{5^5}{2^2}\frac{J^6}{D^3} - 3.2^{11}5^4\frac{LJ}{D^2} - 2^{10}5^5\frac{LJ^3}{D^3} + 2^{20}5^5\frac{L^2}{D^3}\right)\Lambda^2$$
$$+ \left(\frac{55}{2^4} - 41\frac{5^3}{2^2}\frac{J^2}{D} + 307\frac{5^4}{2^4}\frac{J^4}{D^2} + \frac{5^7}{2^3}\frac{J^6}{D^3} + 9.2^3 5^4\frac{LJ}{D^2} - 2^3 5^7\frac{LJ^3}{D^3}\right)\Lambda$$
$$- \left(\frac{3}{2^2} + \frac{5^3}{2^2}\frac{J^2}{D}\right)^3 = 0.$$

I. Let a_1, a_2, a_3, a_4, a_5 be the five roots of a quintic.

Four roots may be selected from these in five ways. By taking these four in different orders, six anharmonic ratios may be formed. Therefore from the five roots thirty anharmonic ratios will be obtained.

It is required to calculate the equation whose roots are these anharmonic ratios.

Let λ_1 be any one of the anharmonic ratios of the roots a_2, a_3, a_4, a_5; then it is known that the remaining five are

$$1-\lambda_1,\ \frac{1}{\lambda_1},\ \frac{\lambda_1-1}{\lambda_1},\ \frac{1}{1-\lambda_1},\ \frac{\lambda_1}{\lambda_1-1}.$$

Now, denoting for brevity a_r-a_s by $\overline{rs}$, where r and s may have any of the values 1, 2, 3, 4, 5, let λ_1 be the anharmonic ratio $\dfrac{(\overline{34})\ (\overline{25})}{(\overline{23})\ (\overline{54})}$, and let λ_2, λ_3, λ_4, λ_5 be obtained from λ_1 by cyclical changes of the suffixes of the roots, so that

$$\lambda_2=\frac{(\overline{45})\ (\overline{31})}{(\overline{34})\ (\overline{15})},\quad \lambda_3=\frac{(\overline{51})\ (\overline{42})}{(\overline{45})\ (\overline{21})},\quad \lambda_4=\frac{(\overline{12})\ (\overline{53})}{(\overline{51})\ (\overline{32})},\quad \lambda_5=\frac{(\overline{23})\ (\overline{14})}{(\overline{12})\ (\overline{43})}.$$

Then each of the thirty anharmonic ratios is a function of one of the five quantities $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$. But these quantities are not all independent. They are subject to the relations

$$\lambda_1\lambda_2+\lambda_4=\lambda_2\lambda_3+\lambda_5=\lambda_3\lambda_4+\lambda_1=\lambda_4\lambda_5+\lambda_2=\lambda_5\lambda_1+\lambda_3=1.$$

These are sufficient to determine any three of them as rational functions of the remaining two. Thus

$$\lambda_3=\frac{1-\lambda_1}{1-\lambda_1\lambda_2},\quad \lambda_4=1-\lambda_1\lambda_2,\quad \lambda_5=\frac{1-\lambda_2}{1-\lambda_1\lambda_2}.$$

Thus the roots of the equation to be calculated are rational functions of two (not any two) of their number.

II. (1) Now, let $\phi(\lambda_1)$ denote

$$(\lambda-\lambda_1)\left(\lambda-(1-\lambda_1)\right)\left(\lambda-\frac{1}{\lambda_1}\right)\left(\lambda-\frac{\lambda_1-1}{\lambda_1}\right)\left(\lambda-\frac{1}{1-\lambda_1}\right)\left(\lambda-\frac{\lambda_1}{\lambda_1-1}\right);$$

the equation which it is required to calculate is

$$\phi(\lambda_1)\,\phi(\lambda_2)\,\phi(\lambda_3)\,\phi(\lambda_4)\,\phi(\lambda_5)=0.$$

Now, $\phi(\lambda_1)=\left(\lambda^2-\lambda+\lambda_1(1-\lambda_1)\right)\left(\lambda^2-\lambda+\dfrac{\lambda_1-1}{\lambda_1^2}\right)\left(\lambda^2-\lambda-\dfrac{\lambda_1}{(\lambda_1-1)^2}\right)$.

Putting $\lambda^2-\lambda=L$,

$$\begin{aligned}\phi(\lambda_1)&=L^3+3L+1+L^2\left\{\frac{(\lambda_1-1)^3-\lambda_1^3-\lambda_1^3(\lambda_1-1)^3}{\lambda_1^2(\lambda_1-1)^2}\right\}\\&=(L+1)^3+L^2\left\{\frac{(\lambda_1-1)^3-\lambda_1^3-\lambda_1^3(\lambda_1-1)^3-3\lambda_1^2(\lambda_1-1)^2}{\lambda_1^2(\lambda_1-1)^2}\right\}\\&=L^2\left\{\frac{(L+1)^3}{L^2}-\frac{(\lambda_1^2-\lambda_1+1)^3}{(\lambda_1^2-\lambda_1)^2}\right\},\end{aligned}$$

or, putting $\dfrac{(L+1)^3}{L^2} = \Lambda$,

$$\phi(\lambda_1) = L^2\left(\Lambda - \frac{(\lambda_1^2-\lambda_1+1)^3}{(\lambda_1^2-\lambda_1)^2}\right).$$

Therefore the equation which it is required to calculate is, omitting the factor L^{10}, of the fifth degree only in Λ, and is

$$\left\{\Lambda - \frac{(\lambda_1^2-\lambda_1+1)^3}{(\lambda_1^2-\lambda_1)^2}\right\}\left\{\Lambda - \frac{(\lambda_2^2-\lambda_2+1)^3}{(\lambda_2^2-\lambda_2)^2}\right\}\left\{\Lambda - \frac{(\lambda_3^2-\lambda_3+1)^3}{(\lambda_3^2-\lambda_3)^2}\right\}$$
$$\left\{\Lambda - \frac{(\lambda_4^2-\lambda_4+1)^3}{(\lambda_4^2-\lambda_4)^2}\right\}\left\{\Lambda - \frac{(\lambda_5^2-\lambda_5+1)^3}{(\lambda_5^2-\lambda_5)^2}\right\} = 0.$$

(2) Now consider the quantity

$$\frac{(\lambda_1^2-\lambda_1+1)^3}{(\lambda_1^2-\lambda_1)^2} = \frac{\left[\left\{\dfrac{(\overline{34})(\overline{25})}{(\overline{23})(\overline{54})}\right\}^2 - \left\{\dfrac{(\overline{34})(\overline{25})}{(\overline{23})(\overline{54})}\right\} + 1\right]^3}{\left[\dfrac{(\overline{34})(\overline{25})}{(23)(\overline{54})}\right]^2\left[\dfrac{(\overline{34})(25)}{(\overline{23})(\overline{54})} - 1\right]^2}$$
$$= \frac{[(\overline{34})^2(\overline{25})^2 - (\overline{34})(\overline{25})(\overline{23})(\overline{54}) + (\overline{23})^2(\overline{54})^2]^3}{(23)^2(\overline{54})^2(\overline{34})^2(25)^2[(\overline{34})(25) - (\overline{23})(\overline{54})]^2}.$$

But
$$(\overline{34})^2(\overline{25})^2 - (\overline{34})(\overline{25})(\overline{23})(\overline{54}) + (\overline{23})^2(\overline{54})^2$$
$$= \{(\overline{34})^2(\overline{25})^2 - (\overline{34})(\overline{25})(\overline{23})(\overline{54})\} + \{(\overline{23})^2(\overline{54})^2 - (\overline{34})(\overline{25})(\overline{23})(\overline{54})\}$$
$$+ \{(\overline{34})(\overline{25})(\overline{23})(\overline{54})\}$$
$$= (\overline{34})(\overline{25})(\overline{35})(\overline{24}) + (\overline{23})(\overline{54})(\overline{24})(\overline{53}) + (\overline{34})(\overline{25})(\overline{23})(\overline{54}),$$

and
$$(\overline{34})(\overline{25}) - (\overline{23})(\overline{54}) = (\overline{35})(\overline{24}).$$

Therefore

$$\frac{(\lambda_1^2-\lambda_1+1)^3}{(\lambda_1^2-\lambda_1)^2} = \frac{[(\overline{23})(\overline{24})(\overline{53})(\overline{54}) + (\overline{24})(\overline{25})(\overline{34})(\overline{35}) + (\overline{25})(\overline{23})(\overline{45})(\overline{43})]^3}{[(23)(\overline{24})(25)(\overline{34})(\overline{35})(\overline{45})]^2},$$

in which form it is seen to be a symmetrical function of $\alpha_2, \alpha_3, \alpha_4, \alpha_5$.

III. The roots of the equation in Λ are not altered when all the roots of the quintic are transformed by the same linear transformation.

Therefore the coefficients of the equation in Λ, which are symmetrical functions of all the five roots, are absolute invariants of the quintic.

The terms which make up the coefficient of Λ^4 can be reduced so as to have for a common denominator the expression

$$B = [(\overline{12})(\overline{13})(\overline{14})(\overline{15})(\overline{23})(\overline{24})(\overline{25})(\overline{34})(\overline{35})(\overline{45})]^2.$$

In like manner, the denominator of the coefficient of Λ^3 will be B^2, and the denominators of the coefficients of Λ^2 and Λ, as well as the denominator of the absolute term, will be B^3.

Thus the weight (*i.e.*, the degree in terms of the roots) of the numerator and denominator of the coefficient of Λ^4 will be 20; that of the numerator and denominator of the coefficient of Λ^3 will be 40; and that of the numerator and denominator of the remaining terms to be calculated will be 60.

As all the denominators are invariants, all the numerators must be invariants, in order that the coefficients may be absolute invariants.

IV. The coefficients will be calculated first in terms of the following three invariants, taken from Salmon's *Lessons on Higher Algebra*, Third Edition, Art. 241:

$$A = \Sigma\,(\overline{12})^4\,(\overline{34})^2\,(\overline{35})^2\,(\overline{45})^2,$$

$$B = [(\overline{12})(\overline{13})(\overline{14})(\overline{15})(\overline{23})(\overline{24})(\overline{25})(\overline{34})(\overline{35})(\overline{45})]^2,$$

$$C = B^2\Sigma\,(\overline{12})^{-4}\,(\overline{34})^{-2}\,(\overline{35})^{-2}\,(\overline{45})^{-2}.$$

The weight of A is 10, of B 20, and of C 30.

Therefore the numerator of the coefficient of Λ^4 will be the sum of numerical multiples of B, A^2; therefore the coefficient itself will be of the form $l+m\,\dfrac{A^2}{B}$.

The numerator of the coefficient of Λ^3 will be the sum of numerical multiples of B^2, BA^2, A^4, CA; therefore the coefficient itself will be of the form $l+m\,\dfrac{A^2}{B}+n\,\dfrac{A^4}{B^2}+q\,\dfrac{CA}{B^2}$.

The numerators of the remaining coefficients to be calculated will be sums of numerical multiples of B^3, B^2A^2, BA^4, A^6, CBA, CA^3, C^2; and these coefficients themselves will therefore take the form

$$l+m\,\frac{A^2}{B}+n\,\frac{A^4}{B^2}+p\,\frac{A^6}{B^3}+q\,\frac{CA}{B^2}+r\,\frac{CA^3}{B^3}+s\,\frac{C^2}{B^3}.$$

V. (1) In order to determine the values of l, m, n, p, q, r, s in each of the coefficients, the quintic will be supposed to take the form

$$x\,(x^2-a^2)\,(x^2-b^2) = 0.$$

In order that this form may be safely employed, it is necessary to show that there is no identical relation between the invariants A, B, C whose weight is 20 or 40 or 60. It will be convenient also, in order to express C in terms of the invariants J, D, L employed in Salmon's *Higher Algebra*, to know that there is no such relation whose weight is 30. But, if an identical relation whose weight is 20 exist, one whose weight is 40 must exist; and, if one whose weight is 30 exist,

then one whose weight is 60 must exist. It will be sufficient, therefore, to show that no identical relation whose weight is 40 or 60 exists. The proofs which are here given do not involve any consideration of the skew invariant I, which vanishes for this form of the quintic, with regard to which Dr. Salmon says, in a note on the Quintic at the end of his *Higher Algebra*: "This form may therefore be safely used in calculating any invariant functions whose order is divisible by 4 and is below 36, since such forms cannot contain I." Now, if the order be below 36, the weight must be below 90, as is the case in all that follows.

Suppose, then, that $a_1 = b$, $a_2 = a$, $a_3 = 0$, $a_4 = -a$, $a_5 = -b$. Then

$$A = 4a^2b^2(a^2+b^2)(3a^4+26a^2b^2+3b^4), \quad B = 16a^6b^6(a^2-b^2)^4,$$

$$C = \left\{ \begin{array}{c} 4a^6b^6(a^2-b^2)^2(a^2+b^2) \\ \times (a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12}) \end{array} \right\}.$$

(2) To show that there exists between the invariants no identical relation whose weight is 40. If such relation exist, it must be possible to find values of l, m, n, q other than zero to satisfy the equation $lB^2+mBA^2+nA^4+qCA = 0$, whatever values a, b may have, *i.e.*

$$l \cdot 2^8a^{12}b^{12}(a^2-b^2)^8 + m \cdot 2^8a^{10}b^{10}(a^2-b^2)^4(a^2+b^2)^2(3a^4+26a^2b^2+3b^4)^2$$
$$+ n \cdot 2^4a^8b^8(a^2+b^2)^4(3a^4+26a^2b^2+3b^4)^4$$
$$+ q \left\{ \begin{array}{c} 2^4a^8b^8(a^2-b^2)^2(a^2+b^2)^2(3a^4+26a^2b^2+3b^4) \\ \times (a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12}) \end{array} \right\} = 0$$

identically. Since this equation is satisfied, whatever a and b may be, it is satisfied in the particular case $a^2 = b^2$; but when $a^2 = b^2$ the first member will not vanish unless $n = 0$.

Putting therefore $n = 0$, and dividing by $2^4a^8b^8(a^2-b^2)^2$, the equation becomes

$$l \cdot 2^4a^4b^4(a^2-b^2)^6 + m \cdot 2^4a^2b^2(a^2-b^2)^2(a^2+b^2)^2(3a^4+26a^2b^2+3b^4)^2$$
$$+ q \left\{ \begin{array}{c} (a^2+b^2)^2(3a^4+26a^2b^2+3b^4) \\ \times (a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12}) \end{array} \right\} = 0.$$

This equation is satisfied for all values of a and b, and therefore when $a^2 = b^2$, therefore $q = 0$.

Putting $q = 0$ and dividing out by $2^4a^2b^2(a^2-b^2)^2$, it becomes

$$l \cdot a^2b^2(a^2-b^2)^4 + m(a^2+b^2)^2(3a^4+26a^2b^2+3b^4)^2 = 0.$$

This must be satisfied when $a^2 = b^2$, therefore $m = 0$.

Putting $m = 0$, it will not be satisfied for all values of a and b unless $l = 0$; therefore the only values of l, m, n, q which satisfy the relation $lB^2+mBA^2+nA^4+qCA = 0$, whatever values a and b may have, are $l = 0$, $m = 0$, $n = 0$, $q = 0$; therefore there exists no identical relation whose weight is 40 between A, B, C.

(3.) To show that there exists between the invariants no identical relation whose weight is 60. If such relation exist, it must be possible to find values of l, m, n, p, q, r, s other than zero to satisfy the equation

$$l\,B^3+m\,A^2B^2+n\,A^4B+p\,A^6+q\,CAB+r\,CA^3+s\,C^2=0,$$

whatever values a, b may have,

i.e.,
$$\begin{aligned}
&l\,.\,2^{12}a^{18}b^{18}\,(a^2-b^2)^{12}+m\,.\,2^{12}a^{16}b^{16}(a^2-b^2)^8\,(a^2+b^2)^2\,(3a^4+26a^2b^2+3b^4)^2\\
&\quad+n\,.\,2^{12}a^{14}b^{14}\,(a^2-b^2)^4\,(a^2+b^2)^4\,(3a^4+26a^2b^2+3b^4)^4\\
&\quad+p\,.\,2^{12}a^{12}b^{12}\,(a^2+b^2)^6\,(3a^4+26a^2b^2+3b^4)^6\\
&\quad+q\left\{\begin{array}{c}2^8a^{14}b^{14}\,(a^2-b^2)^6\,(a^2+b^2)^2\,(3a^4+26a^2b^2+3b^4)\\ \times\,(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})\end{array}\right\}\\
&\quad+r\left\{\begin{array}{c}2^8a^{12}b^{12}\,(a^2-b^2)^2\,(a^2+b^2)^4\,(3a^4+26a^2b^2+3b^4)^3\\ \times\,(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})\end{array}\right\}\\
&\quad+s\left\{\begin{array}{c}2^4a^{12}b^{12}\,(a^2-b^2)^4\,(a^2+b^2)^2\\ \times\,(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})^2\end{array}\right\}\\
&\qquad\qquad=0 \text{ identically};
\end{aligned}$$

since this equation is satisfied, whatever a and b may be, it is satisfied in the particular case $a^2=b^2$; but, when $a^2=b^2$, the first member will not vanish unless $p=0$: therefore $p=0$.

Now divide out by $2^4a^{12}b^{12}\,(a^2-b^2)^2$ after putting $p=0$ in the above, and it becomes

$$\begin{aligned}
&l\,.\,2^8a^6b^6\,(a^2-b^2)^{10}+m\,.\,2^8a^4b^4\,(a^2-b^2)^6\,(a^2+b^2)^2\,(3a^4+26a^2b^2+3b^4)^2\\
&\quad+n\,.\,2^8a^2b^2\,(a^2-b^2)^2\,(a^2+b^2)^4\,(3a^4+26a^2b^2+3b^4)^4\\
&\quad+q\left\{\begin{array}{c}2^4a^2b^2\,(a^2-b^2)^4\,(a^2+b^2)^2\,(3a^4+26a^2b^2+3b^4)\\ \times\,(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})\end{array}\right\}\\
&\quad+r\left\{\begin{array}{c}2^4\,(a^2+b^2)^4\,(3a^4+26a^2b^2+3b^4)^3\\ \times\,(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})\end{array}\right\}\\
&\quad+s\left\{\begin{array}{c}(a^2-b^2)^2\,(a^2+b^2)^2\\ \times\,(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})^2\end{array}\right\}\\
&\qquad\qquad=0.
\end{aligned}$$

This equation must be true when $a^2=b^2$, therefore $r=0$.

Dividing by $(a^2-b^2)^2$, it becomes, after putting $r=0$,

$$\begin{aligned}
&l\,.\,2^8a^6b^6\,(a^2-b^2)^8+m\,.\,2^8a^4b^4\,(a^2-b^2)^4\,(a^2+b^2)^2\,(3a^4+26a^2b^2+3b^4)^2\\
&\quad+n\,.\,2^8a^2b^2\,(a^2+b^2)^4\,(3a^4+26a^2b^2+3b^4)^4\\
&\quad+q\left\{\begin{array}{c}2^4a^2b^2\,(a^2-b^2)^2\,(a^2+b^2)^2\,(3a^4+26a^2b^2+3b^4)\\ \times\,(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})\end{array}\right\}\\
&\quad+s\,.\,(a^2+b^2)^2\,(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})^2\\
&\qquad\qquad=0.
\end{aligned}$$

This must be true when $a^2=-b^2$, therefore $l=0$.

Putting $l = 0$, and dividing by $(a^2+b^2)^2$, it becomes

$$m \cdot 2^8 a^4 b^4 (a^2-b^2)^4 (3a^4+26a^2b^2+3b^4)^2 + n \cdot 2^8 a^2 b^2 (a^2+b^2)^2 (3a^4+26a^2b^2+3b^4)^4$$
$$+q \left\{ \begin{array}{c} 2^4 a^2 b^2 (a^2-b^2)^2 (3a^4+26a^2b^2+3b^4) \\ \times (a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12}) \end{array} \right\}$$
$$+s (a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})^2 = 0.$$

This is true when $a^2 = 0$, therefore $s = 0$.

Putting $s = 0$ and dividing by $2^4 a^2 b^2$, it becomes

$$m \cdot 2^4 a^2 b^2 (a^2-b^2)^4 (3a^4+26a^2b^2+3b^4)^2 + n \cdot 2^4 (a^2+b^2)^2 (3a^4+26a^2b^2+3b^4)^4$$
$$+q \left\{ \begin{array}{c} (a^2-b^2)^2 (3a^4+26a^2b^2+3b^4) \\ \times (a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12}) \end{array} \right\} = 0.$$

This is true when $a^2 = b^2$, therefore $n = 0$.

Putting $n = 0$ and dividing out by $(a^2-b^2)^2$, it becomes

$$m \cdot 2^4 a^2 b^2 (a^2-b^2)^2 (3a^4+26a^2b^2+3b^4)^2$$
$$+q \left\{ \begin{array}{c} (3a^4+26a^2b^2+3b^4) \\ \times (a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12}) \end{array} \right\} = 0.$$

This is true when $a^2 = b^2$, therefore $q = 0$.

Putting $q = 0$, the equation will not be satisfied for all values of a and b, unless also $m = 0$.

Therefore the only values of l, m, n, p, q, r, s which make

$$lB^3+mA^2B^2+nA^4B+pA^6+qCAB+rCA^3+sC^2 = 0,$$

whatever values a and b may have, are $l = 0$, $m = 0$, $n = 0$, $p = 0$, $q = 0$, $r = 0$, $s = 0$.

Therefore there is no identical relation whose weight is 60 between A, B, C.

VI. Also, since $a_1 = b$, $a_2 = a$, $a_3 = 0$, $a_4 = -a$, $a_5 = -b$,

$$\overline{12} = b-a; \quad \overline{13} = b; \quad \overline{14} = b+a; \quad \overline{15} = 2b;$$
$$\overline{23} = a; \quad \overline{24} = 2a; \quad \overline{25} = a+b;$$
$$\overline{34} = a; \quad \overline{35} = b;$$
$$\overline{45} = b-a;$$

$$\lambda_1 = \lambda_5 = \frac{a+b}{a-b}; \quad \lambda_2 = \lambda_4 = \frac{a-b}{2a}; \quad \lambda_3 = -\frac{4ab}{(a-b)^2};$$

$$\frac{(\lambda_1^2-\lambda_1+1)^3}{(\lambda_1^2-\lambda_1)^2} = \frac{(a^2+3b^2)^3}{4b^2(a^2-b^2)^2} = \frac{(\lambda_5^2-\lambda_5+1)^3}{(\lambda_5^2-\lambda_5)^2},$$

$$\frac{(\lambda_2^2-\lambda_2+1)^3}{(\lambda_2^2-\lambda_2)} = \frac{(3a^2+b^2)^3}{4a^2(a^2-b^2)^2} = \frac{(\lambda_4^2-\lambda_4+1)^3}{(\lambda_4^2-\lambda_4)^2},$$

$$\frac{(\lambda_3^2-\lambda_3+1)^3}{(\lambda_3-\lambda_3)^2} = \frac{(a^4+14a^2b^2+b^4)^3}{16a^2b^2(a^2-b^2)^4}.$$

Therefore the equation in Λ is

$$\left[\Lambda - \frac{(a^2+3b^2)^3}{4b^2\,(a^2-b^2)^2}\right]^2 \left[\Lambda - \frac{(3a^2+b^2)^3}{4a^2\,(a^2-b^2)^2}\right]^2 \left[\Lambda - \frac{(a^4+14a^2b^2+b^4)^3}{16a^2b^2\,(a^2-b^2)^4}\right] = 0.$$

This is $$\Lambda^5 - l_1\Lambda^4 + l_2\Lambda^3 - l_3\Lambda^2 + l_4\Lambda - l_5 = 0,$$

where

$$l_1 = \frac{a^2\,(a^2+3b^2)^3 + b^2\,(3a^2+b^2)^3}{2a^2b^2\,(a^2-b^2)^2} + \frac{(a^4+14a^2b^2+b^4)^3}{16a^2b^2\,(a^2-b^2)^4},$$

$$l_2 = \frac{b^4\,(3a^2+b^2)^6 + 4a^2b^2\,(a^2+3b^2)^3\,(3a^2+b^2)^3 + a^4\,(a^2+3b^2)^6}{16a^4b^4\,(a^2-b^2)^4}$$

$$+ \left\{\frac{(a^4+14a^2b^2+b^4)^3}{16a^2b^2\,(a^2-b^2)^4} \cdot \frac{a^2\,(a^2+3b^2)^3 + b^2\,(3a^2+b^2)^3}{2a^2b^2\,(a^2-b^2)^2}\right\},$$

$$l_3 = \frac{(a^2+3b^2)^3\,(3a^2+b^2)^3\,\{b^2\,(3a^2+b^2)^3 + a^2\,(a^2+3b^2)^3\}}{32a^4b^4\,(a^2-b^2)^6}$$

$$+ \left\{\frac{(a^4+14a^2b^2+b^4)^3}{16a^2b^2\,(a^2-b^2)^4} \cdot \frac{b^4(3a^2+b^2)^6 + 4a^2b^2\,(a^2+3b^2)^3\,(3a^2+b^2)^3 + a^4\,(a^2+3b^2)^6}{16a^4b^4\,(a^2-b^2)^4}\right\},$$

$$l_4 = \frac{(a^2+3b^2)^6\,(3a^2+b^2)^6}{256\,a^4b^4\,(a^2-b^2)^8}$$

$$+ \left\{\frac{(a^4+14a^2b^2+b^4)^3}{16a^2b^2\,(a^2-b^2)^4} \cdot \frac{(a^2+3b^2)^3\,(3a^2+b^2)^3}{32a^4b^4\,(a^2-b^2)^6}\,[b^2(3a^2+b^2)^3 + a^2(a^2+3b^2)^3]\right\},$$

$$l_5 = \frac{(a^2+3b^2)^6\,(3a^2+b^2)^6\,(a^4+14a^2b^2+b^4)^3}{256a^4b^4\,(a^2-b^2)^8 \cdot 16a^2b^2\,(a^2-b^2)^4}.$$

VII. (1) to calculate l_1.

Let $$l_1 = l + m\frac{A^2}{B}.$$

Then $$\frac{a^2\,(a^2+3b^2)^3 + b^2\,(3a^2+b^2)^3}{2a^2b^2\,(a^2-b^2)^2} + \frac{(a^4+14a^2b^2+b^4)^3}{16a^2b^2\,(a^2-b^2)^4}$$

$$= l + m\,\frac{16a^4b^4\,(3a^6+29a^4b^2+29a^2b^4+3b^6)^2}{16a^6b^6\,(a^2-b^2)^4}.$$

Multiply both sides by $16a^2b^2\,(a^2-b^2)^4$;

therefore $$8\,(a^2-b^2)^2\,[a^2\,(a^2+3b^2)^3 + b^2\,(3a^2+b^2)^3] + (a^4+14a^2b^2+b^4)^3$$

$$= l \cdot 16a^2b^2\,(a^2-b^2)^4 + 16m\,(3a^6+29a^4b^2+29a^2b^4+3b^6)^2.$$

The value of l may in this case be very conveniently determined by putting $a^2 = -b^2$ on both sides. This will give one equation containing l and numbers only.

The value of m may in like manner be determined by putting $a^2 = b^2$ on both sides. A method analogous to this will in some cases be more convenient than the one adopted in this paper; but the method given here leads to equations which are well adapted to verify the values of l, m, ... which are found.

It will therefore be followed throughout.

It consists in equating the coefficients of the different powers of a and b on both sides.

The above equation* is

$$\left.\begin{aligned} &8\,(a^{12}+34a^{10}b^2-17a^8b^4-36a^6b^6-\ldots)\\ &+\,(a^{12}+42a^{10}b^2+591a^8b^4+2828a^6b^6+\ldots)\end{aligned}\right\}$$
$$= \left\{\begin{aligned} &16l\,.\,a^2b^2\,(a^8-4a^6b^2+6a^4b^4-\ldots)\\ &+16m\,(9a^{12}+174a^{10}b^2+1015a^8b^4+1700a^6b^6+\ldots).\end{aligned}\right.$$

Equating coefficients of the different powers of a, b,

$$\begin{aligned} 9 &= 9\,(16m),\\ 314 &= 174\,(16m)+(16l),\\ 455 &= 1015\,(16m)-4\,(16l),\\ 2540 &= 1700\,(16m)+6\,(16l). \end{aligned}$$

From the first and second equations, $16m = 1$, $16l = 140$.

The next two equations may be used to verify these values of l and m.

Therefore
$$l_1 = \frac{140}{2^4} + \frac{1}{2^4}\cdot\frac{A^2}{B}.$$

(2.) To calculate l_2.

$$l_2 = \frac{a^{16}+126a^{14}b^2+1944a^{12}b^4+5922a^{10}b^6+8590a^8b^8+\ldots}{16a^4b^4\,(a^2-b^2)^4}$$
$$+\frac{a^{20}+78a^{18}b^2+2157a^{16}b^4+26408a^{14}b^6+135826a^{12}b^8+195348a^{10}b^{10}+\ldots}{32a^4b^4\,(a^2-b^2)^6}$$
$$= l+m\,\frac{A^2}{B}+n\,\frac{A^4}{B^2}+q\,\frac{CA}{B^2}$$
$$= l+m\,.\,\frac{16a^4b^4\,(3a^6+29a^4b^2+29a^2b^4+3b^6)^2}{16a^6b^6\,(a^2-b^2)^4}$$
$$+n\,.\,\frac{256a^8b^8\,(3a^6+29a^4b^2+29a^2b^4+3b^6)^4}{256a^{12}b^{12}\,(a^2-b^2)^8}$$
$$+q\,.\,\frac{\left\{\begin{aligned}&16a^8b^8\,(a^2-b^2)^2\,(a^2+b^2)(3a^6+29a^4b^2+29a^2b^4+3b^6)\\ &\times(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})\end{aligned}\right\}}{256a^{12}b^{12}\,(a^2-b^2)^8}$$

Let the second and fourth of the above equal quantities be equated.

It may be observed at once that, if both sides of the equation obtained are multiplied by $(a^2-b^2)^8$, one side will vanish when $a^2 = b^2$, but the other will not unless $n = 0$. Therefore $n = 0$, since the equation is to be identically true whatever a and b may be.

Now, after putting $n = 0$, multiply the second and fourth of the above equal quantities by $32a^4b^4\,(a^2-b^2)^6$ and equate the results.

---

* Throughout this paper, when an expression is symmetrical with regard to a and b, it will not in general be completely written down, but only those terms which it is necessary to know in order to be able to write down the rest, which will be indicated by dots.

$$\therefore\ (2a^{20}+248a^{18}b^2+3386a^{16}b^4+4320a^{14}b^6-2620a^{12}b^8-10672a^{10}b^{10}-\ldots)$$
$$+(a^{20}+78a^{18}b^2+2157a^{16}b^4+26408a^{14}b^6+135826a^{12}b^8+195348a^{10}b^{10}+\ldots)$$
$$=(32l)\,(a^{16}b^4-6a^{14}b^6+15a^{12}b^8-20a^{10}b^{10}-\ldots)$$
$$+(32m)\,(9a^{18}b^2+156a^{16}b^4+676a^{14}b^6-156a^{12}b^8-1370a^{10}b^{10}-\ldots)$$
$$+(2q)(3a^{20}+110a^{18}b^2+1319a^{16}b^4+17384a^{14}b^6+129750a^{12}b^8+227156a^{10}b^{10}+\ldots).$$

Hence, to determine l, m, q, there are the equations

$$3=3\,(2q),$$
$$326=110\,(2q)+9\,(32m),$$
$$5543=1319\,(2q)+156\,(32m)+(32l),$$

together with three other equations which may be used to verify the values of l, m, q obtained from these.

Solving the equations, $2q=1$,
$32m=24$,
$32l=480$,

and it was shown above that

$$n=0;$$

therefore
$$l_2=\frac{480}{2^5}+\frac{24}{2^5}\frac{A^2}{B}+\frac{1}{2}\frac{CA}{B^2}.$$

(3) To calculate l_3,

$$l_3=\frac{(a^2+3b^2)^3\,(3a^2+b^2)^3\,\{b^2\,(3a^2+b^2)^3+a^2\,(a^2+3b^2)^3\}}{32a^4b^4\,(a^2-b^2)^6}$$
$$+\frac{(a^4+14a^2b^2+b^4)^3\{a^4(a^2+3b^2)^6+4a^2b^2(a^2+3b^2)^3(3a^2+b^2)^3+b^4(3a^2+b^2)^6\}}{256a^6b^6\,(a^2-b^2)^8}$$
$$=l+m\frac{A^2}{B}+n\frac{A^4}{B^2}+p\frac{A^6}{B^6}+q\frac{CA}{B^2}+r\frac{CA^3}{B^3}+s\frac{C^2}{B^3}$$
$$=l+m\frac{16a^4b^4\,(3a^6+29a^4b^2+29a^2b^4+3b^6)^2}{16a^6b^6\,(a^2-b^2)^4}$$
$$+n\frac{256a^8b^8\,(3a^6+29a^4b^2+29a^2b^4+3b^6)^4}{256a^{12}b^{12}\,(a^2-b^2)^8}$$
$$+p\frac{4096a^{12}b^{12}\,(3a^6+29a^4b^2+29a^2b^4+3b^6)^6}{4096a^{18}b^{18}\,(a^2-b^2)^{12}}$$
$$+q\frac{\left\{\begin{array}{c}16a^8b^8\,(a^2-b^2)^2\,(a^2+b^2)\,(3a^6+29a^4b^2+29a^2b^4+3b^6)\\ \times(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})\end{array}\right\}}{256a^{12}b^{12}\,(a^2-b^2)^8}$$
$$+r\frac{\left\{\begin{array}{c}256a^{12}b^{12}\,(a^2-b^2)^2\,(a^2+b^2)\,(3a^6+29a^4b^2+29a^2b^4+3b^6)^3\\ \times(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})\end{array}\right\}}{4096a^{18}b^{18}\,(a^2-b^2)^{12}}$$
$$+s\frac{\left\{\begin{array}{c}16a^{12}b^{12}\,(a^2-b^2)^4\,(a^2+b^2)^2\\ \times(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+143a^4b^8+26a^2b^{10}+b^{12})^2\end{array}\right\}}{4096a^{18}b^{18}\,(a^2-b^2)^{12}}$$

Let the second and fourth of the above equal quantities be equated. It may be observed at once that, if both sides of the equation obtained are multiplied by $(a^2-b^2)^{12}$, one side will vanish when $a^2=b^2$, but the other will not unless $p=0$: therefore $p=0$. Putting $p=0$ in the above, and multiplying both sides by $(a^2-b^2)^{10}$, by the same argument $r=0$.

After putting $p=0$, $r=0$, multiply both sides by $256a^6b^6(a^2-b^2)^8$; the equation becomes

$$\begin{aligned}
&8a^2b^2\,(27a^{24}+1188a^{22}b^2+9702a^{20}b^4+29332a^{18}b^6\\
&\qquad\qquad +26485a^{16}b^8-31544a^{14}b^{10}-70380a^{12}b^{12}-\ldots)\\
&\quad+(a^{28}+168a^{26}b^2+7827a^{24}b^4+164864a^{22}b^6+1763137a^{20}b^8\\
&\qquad\qquad +9438744a^{18}b^{10}+23228971a^{16}b^{12}+31455872a^{14}b^{14}+\ldots)\\
&=(l\,.\,256)\;a^6b^6\,(a^{16}-8a^{14}b^2+28a^{12}b^4-56a^{10}b^6+70a^8b^8-\ldots)\\
&\quad+(m\,.\,256)\;a^4b^4\,(9a^{20}+138a^{18}b^2\\
&\qquad\qquad +373a^{16}b^4-1352a^{14}b^6-382a^{12}b^8+2428a^{10}b^{10}-\ldots)\\
&\quad+(n\,.\,256)\;a^2b^2\,(81a^{24}+3132a^{22}b^2+48546a^{20}b^4+383820a^{18}b^6\\
&\qquad\qquad +1640095a^{16}b^8+3807352a^{14}b^{10}+5011164a^{12}b^{12}+\ldots)\\
&\quad+(q\,.\,16)\;a^2b^2\,(3a^{24}+104a^{22}b^2+1102a^{20}b^4+14856a^{18}b^6\\
&\qquad\qquad +96301a^{16}b^8-14960a^{14}b^{10}-194812a^{12}b^{12}-\ldots)\\
&\quad+s\,.\,(a^{28}+54a^{26}b^2+1067a^{24}b^4+16924a^{22}b^6+246905a^{20}b^8\\
&\qquad\qquad +1528746a^{18}b^{10}+16529243a^{16}b^{12}+30462984a^{14}b^{14}+\ldots).
\end{aligned}$$

Equating the coefficients of the different powers of a, b, there arise the equations

$$\begin{aligned}
1&=s,\\
384&=54s+3\,(16q)+81\,(256n),\\
17331&=1067s+104\,(16q)+3132\,(256n)+9\,(256m),\\
242480&=16924s+1102\,(16q)+48546\,(256n)+138\,(256m)+(256l),\\
1997793&=246905s+14856\,(16q)+383820\,(256n)+373\,(256m)-8(256l),
\end{aligned}$$

and three other equations which may be used for verification.

Their solutions are

$$\begin{aligned}
256l&=2720,\\
256m&=464,\\
256n&=2,\\
16q&=56,\\
s&=1.
\end{aligned}$$

Therefore

$$l_3=\frac{2720}{2^8}+\frac{464}{2^8}\cdot\frac{A^2}{B}+\frac{2}{2^8}\cdot\frac{A^4}{B^2}+\frac{56}{2^4}\cdot\frac{CA}{B^2}+\frac{C^2}{B^3}.$$

(4) To calculate l_4,

$$l_4 = \frac{(a^2+3b^2)^6\,(3a^2+b^2)^6}{256a^4b^4\,(a^2-b^2)^8} + \frac{(a^4+14a^2b^2+b^4)^3\,(a^2+3b^2)^3\,(3a^2+b^2)^3\,\{b^2\,(3a^2+b^2)^3+a^2\,(a^2+3b^2)^3\}}{512a^6b^6\,(a^2-b^2)^{10}}$$

$$= l+m\,\frac{A^2}{B}+n\,\frac{A^4}{B^2}+p\,\frac{A^6}{B^3}+q\,\frac{CA}{B^2}+r\,\frac{CA^3}{B^3}+s\,\frac{C^2}{B^3}$$

= {the fourth of the four equal quantities written in VII. (3) on page 90}.

Multiplying the second and the fourth of the above equal quantities by $(a^2-b^2)^{12}$, and observing that the two expressions so obtained are identically equal whatever a and b may be, and therefore when $a^2=b^2$, it follows that $p=0$.

Putting, then, $p=0$, and multiplying the second and fourth of the above equal quantities by $512a^6b^6\,(a^2-b^2)^{10}$, it follows that

$$\begin{aligned}
&2a^2b^2\,[729a^{28}+13122a^{26}b^2+97443a^{24}b^4+375732a^{22}b^6+747009a^{20}b^8\\
&\qquad +484830a^{18}b^{10}-841085a^{16}b^{12}-1755560a^{14}b^{14}-\ldots]\\
&+[27a^{32}+2376a^{30}b^2+80280a^{28}b^4+1373464a^{26}b^6+13034580a^{24}b^8\\
&\qquad +71252232a^{22}b^{10}+232461992a^{20}b^{12}+464685208a^{18}b^{14}\\
&\qquad +581703330a^{16}b^{16}+\ldots]\\
=\;& 512l\,.\,a^6b^6\,[a^{20}-10a^{18}b^2+45a^{16}b^4-120a^{14}b^6+210a^{12}b^8-252a^{10}b^{10}+\ldots]\\
&+512m\,.\,a^4b^4\,[9a^{24}+120a^{22}b^2+106a^{20}b^4-1960a^{18}b^6+2695a^{16}b^8\\
&\qquad +1840a^{14}b^{10}-5620a^{12}b^{12}+\ldots]\\
&+512n\,.\,a^2b^2\,[81a^{28}+2970a^{26}b^2+42363a^{24}b^4+289860a^{22}b^6+921001a^{20}b^8\\
&\qquad +910982a^{18}b^{10}-963445a^{16}b^{12}-2407624a^{14}b^{14}-\ldots]\\
&+32q\,.\,a^2b^2\,[3a^{28}+98a^{26}b^2+897a^{24}b^4+12756a^{22}b^6+67691a^{20}b^8\\
&\qquad -192706a^{18}b^{10}-68591a^{16}b^{12}+359704a^{14}b^{14}-\ldots]\\
&+32r\,[27a^{32}+1512a^{30}b^2+34056a^{28}b^4+502712a^{26}b^6+5721396a^{24}b^8\\
&\qquad +44620136a^{22}b^{10}+203299896a^{20}b^{12}+491746552a^{18}b^{14}\\
&\qquad +655631074a^{16}b^{16}+\ldots]\\
&+2s\,[a^{32}+52a^{30}b^2+960a^{28}b^4+14844a^{26}b^6+214124a^{24}b^8+1051860a^{22}b^{10}\\
&\qquad +13718656a^{20}b^{12}-1066756a^{18}b^{14}-27867482a^{16}b^{16}-\ldots].
\end{aligned}$$

Hence, equating the coefficients of the different powers of a, b,

$$\begin{aligned}
27 &= (2s)+27\,(32r),\\
3834 &= 52\,(2s)+1512\,(32r)+3\,(32q)+81\,(512n),\\
106524 &= 960\,(2s)+34056\,(32r)+98\,(32q)+2970\,(512n)+9\,(512m),\\
1568350 &= 14844\,(2s)+502712\,(32r)+897\,(32q)+42363\,(512n)\\
&\qquad +120\,(512m)+(512l),
\end{aligned}$$

$$13786044 = 214124\,(2s) + 5721396\,(32r) + 12756\,(32q) + 289860\,(512n) + 106\,(512m) - 10\,(512l),$$

$$72746250 = 1051860\,(2s) + 44620136\,(32r) + 67691\,(32q) + 921001\,(512n) - 1960\,(512m) + 45\,(512l),$$

and three other equations which may be used for verification.

Their solutions are

$$512l = 1760,$$
$$512m = -1456,$$
$$512n = 30,$$
$$32q = -36,$$
$$32r = 1,$$
$$2s = 0,$$

therefore $$l_4 = \frac{1760}{2^9} - \frac{1456}{2^9}\frac{A^2}{B} + \frac{30}{2^9}\frac{A^4}{B^2} - \frac{36}{2^5}\frac{CA}{B^2} + \frac{1}{2^5}\frac{CA^3}{B^3}.$$

(5) To calculate l_5.

This may be done in the same way as l_3 and l_4. But, as it will take the form of an exact cube when expressed in terms of the invariants, it will be much shorter to commence by expressing its cube root in terms of the invariants.

$$(l_5)^{\frac{1}{3}} = \frac{(a^2+3b^2)^2\,(3a^2+b^2)^2\,(a^4+14a^2b^2+b^4)}{16a^2b^2\,(a^2-b^2)^4}.$$

Now, if, as has just been stated, l_5 can be expressed as the cube of a fraction the weight of whose numerator is 20 and equal to the weight of the denominator; then $(l_5)^{\frac{1}{3}}$ can be expressed in terms of the invariants as a fraction the weight of whose numerator is 20 and equal to the weight of the denominator.

Assume therefore, if possible,

$$(l_5)^{\frac{1}{3}} = \frac{(a^2+3b^2)^2\,(3a^2+b^2)^2\,(a^4+14a^2b^2+b^4)}{16a^2b^2\,(a^2-b^2)^4} = l + m\frac{A^2}{B}$$
$$= l + m\,\frac{16a^4b^4\,(3a^6+29a^4b^2+29a^2b^4+3b^6)^2}{16a^6b^6\,(a^2-b^2)^4}.$$

Multiplying both sides by $16a^2b^2\,(a^2-b^2)^4$, there results

$$(a^2+3b^2)^2\,(3a^2+b^2)^2\,(a^4+14a^2b^2+b^4) = l\,.\,16a^2b^2\,(a^2-b^2)^4 + m\,.\,16\,(3a^6+29a^4b^2+29a^2b^4+3b^6)^2,$$

therefore $$9a^{12}+186a^{10}b^2+967a^8b^4+1772a^6b^6+967a^4b^8+186a^2b^{10}+9b^{12}$$
$$= 16l\,.\,a^2b^2\,(a^8-4a^6b^2+6a^4b^4-4a^2b^6+b^8)$$
$$+16m\,(9a^{12}+174a^{10}b^2+1015a^8b^4+1700a^6b^6+1015a^4b^8+174a^2b^{10}+9b^{12}).$$

Equating coefficients of the different powers of a and b,

$$9 = 9\,(16m),$$
$$186 = 174\,(16m) + (16l),$$
$$967 = 1015\,(16m) - 4\,(16l),$$
$$1772 = 1700\,(16m) + 6\,(16l).$$

All the equations are written down in this case, because the expression of $(l_5)^{\frac{1}{5}}$ as a rational function of the invariants is not yet known to be possible.

From the first two of the above equations, $16m = 1$, $16l = 12$.

Since these values satisfy the remaining equations, the expression of $(l_5)^{\frac{1}{5}}$ as a rational function of the invariants is possible; and thus the truth of the statement above is verified.

Therefore
$$(l_5)^{\frac{1}{5}} = \frac{12}{2^4} + \frac{1}{2^4}\,\frac{A^2}{B},$$

therefore
$$l_5 = \left\{\frac{12}{2^4} + \frac{1}{2^4}\,\frac{A^2}{B}\right\}^5.$$

(6) Therefore the equation in Λ becomes

$$\Lambda^5 - \left(\frac{140}{2^4} + \frac{1}{2^4}\,\frac{A^2}{B}\right)\Lambda^4 + \left(\frac{480}{2^5} + \frac{24}{2^5}\,\frac{A^2}{B} + \frac{1}{2}\,\frac{CA}{B^2}\right)\Lambda^3$$
$$- \left(\frac{2720}{2^8} + \frac{464}{2^8}\,\frac{A^2}{B} + \frac{2}{2^8}\,\frac{A^4}{B^2} + \frac{56}{2^4}\,\frac{CA}{B^2} + \frac{C^2}{B^3}\right)\Lambda^2$$
$$+ \left(\frac{1760}{2^9} - \frac{1456}{2^9}\,\frac{A^2}{B} + \frac{30}{2^9}\,\frac{A^4}{B^2} - \frac{36}{2^5}\,\frac{CA}{B^2} + \frac{1}{2^5}\,\frac{CA^3}{B^3}\right)\Lambda$$
$$- \left(\frac{12}{2^4} + \frac{1}{2^4}\,\frac{A^2}{B}\right)^3 = 0.$$

VIII. In order to express the coefficients of the equation in Λ in terms of the invariants J, K, L and the discriminant $D = J^2 - 128K$ employed in Salmon's *Higher Algebra*, it will be necessary to calculate them for the equation $x\,(x^2-a^2)\,(x^2-b^2) = 0$.

For this form, the expression for J given in Art. 225 of Salmon's *Higher Algebra* reduces to the two terms

$$16ace^2 + 48c^3e;$$

but
$$a = 1,\ c = -\frac{a^2+b^2}{10},\ e = \frac{a^2b^2}{5};$$

therefore
$$J = -\frac{4a^2b^2\,(a^2+b^2)\,(3a^4+26a^2b^2+3b^4)}{1250} = -\frac{A}{1250}.$$

The expression for K in Art. 226 reduces to the terms

$$-2a^3e^5 + 22a^2c^2e^4 - 38ac^4e^3 + 18c^6e^2$$
$$= \frac{a^4b^4}{5^3.10^5}(9a^{12} - 326a^{10}b^2 + 3015a^8b^4 - 1300a^6b^6 + 3015a^4b^8 - 326a^2b^{10} + 9b^{12});$$

therefore $$D = J^2 - 128K = \frac{1}{5^5} 16a^6b^6(a^2-b^2)^4 = \frac{B}{5^5}.$$

The expression for L in Art. 227 reduces to

$$4a^4ce^7 - 16a^3c^3e^6 + 24a^2c^5e^5 - 16ac^7e^4 + 4c^9e^3,$$

the last term being picked out because complementary to the term $4b^3d^9$, and therefore implicitly contained in the expression $4(b^3d^9)$ which stands for $4b^3d^9 + 4c^9e^3$ in the notation of that article.

In this expression for L, first put $a = 1$, leaving the substitution of the values of c and e till the next step.

Thus $$L = 4ce^3(e-c^2)^4.$$

Now, substituting for c and e their values,

$$L = -\frac{1}{2^7 5^{12}} a^6b^6(a^2+b^2)(a^4-18a^2b^2+b^4)^4.$$

Now C, being an invariant whose weight is 30, can be expressed in the form $$\alpha J^3 + \beta JD + \gamma L;$$

therefore

$$\begin{aligned} 4a^6b^6(a^2-b^2)^2(a^2+b^2)(a^{12}+26a^{10}b^2+143a^8b^4+3756a^6b^6+\ldots) \\ = \alpha\left(-\frac{2^3}{5^{12}}\right) a^6b^6(a^2+b^2)^3(3a^4+26a^2b^2+3b^4)^3 \\ +\beta\left(-\frac{2^5}{5^9}\right) a^8b^8(a^2+b^2)(3a^4+26a^2b^2+3b^4)(a^2-b^2)^4 \\ +\gamma\left(-\frac{1}{2^7 5^{12}}\right) a^6b^6(a^2+b^2)(a^4-18a^2b^2+b^4)^4. \end{aligned}$$

Dividing both sides of the equation by $a^6b^6(a^2+b^2)$, it assumes after reduction the form

$$\begin{aligned} 4(a^{16}+24a^{14}b^2+92a^{12}b^4+3496a^{10}b^6-7226a^8b^8+\ldots) \\ = \alpha\left(-\frac{2^3}{5^{12}}\right)(27a^{16}+756a^{14}b^2+7596a^{12}b^4+32012a^{10}b^6+50290a^8b^8+\ldots) \\ +\beta\left(-\frac{2^5}{5^9}\right)(\quad 3a^{14}b^2+14a^{12}b^4-83a^{10}b^6+132a^8b^8+\ldots) \\ +\gamma\left(-\frac{1}{2^7 5^{12}}\right)(a^{16}-72a^{14}b^2+1948a^{12}b^4-23544a^{10}b^6+108870a^8b^8+\ldots). \end{aligned}$$

Equating coefficients of different powers of a and b,

$$\begin{aligned} 4 &= \alpha\left(-\frac{2^3}{5^{12}}\right)27 + \gamma\left(-\frac{1}{2^7 5^{12}}\right), \\ 96 &= \alpha\left(-\frac{2^3}{5^{12}}\right)756 + \beta\left(-\frac{2^5}{5^9}\right)3 + \gamma\left(-\frac{1}{2^7 5^{12}}\right)(-72), \\ 368 &= \alpha\left(-\frac{2^3}{5^{12}}\right)7596 + \beta\left(-\frac{2^5}{5^9}\right)14 + \gamma\left(-\frac{1}{2^7 5^{12}}\right)1948, \end{aligned}$$

$$13984 = \alpha\left(-\frac{2^3}{5^{12}}\right) 32012 + \beta\left(-\frac{2^5}{5^9}\right)(-83) + \gamma\left(-\frac{1}{2^7 5^{12}}\right)(-23544)$$

$$-28904 = \alpha\left(-\frac{2^3}{5^{12}}\right) 50290 + \beta\left(-\frac{2^5}{5^9}\right) 132 + \gamma\left(-\frac{1}{2^7 5^{12}}\right) 108870.$$

The solutions of the first three equations are

$$\alpha\left(-\frac{2^3}{5^{12}}\right) = \frac{2^2}{5^2};\quad \beta\left(-\frac{2^5}{5^9}\right) = -2^4;\quad \gamma\left(-\frac{1}{2^7 5^{12}}\right) = -\frac{2^3}{5^2},$$

and these values satisfy the last two equations.

Therefore $$\alpha = -\frac{5^{10}}{2};\quad \beta = \frac{5^9}{2};\quad \gamma = 2^{10}.5^{10};$$

therefore* $$C = \frac{1}{2}.5^{10}\left(2^{11}L - J^3 + \frac{1}{5}JD\right).$$

Making these substitutions, the equation becomes

$$\Lambda^5 - \left(\frac{35}{2^2} + \frac{5^3}{2^2}\frac{J^2}{D}\right)\Lambda^4 + \left(15 + \frac{5^4}{2}\frac{J^2}{D} + \frac{5^4}{2}\frac{J^4}{D^2} - \frac{2^{10}5^4 JL}{D^2}\right)\Lambda^3$$
$$-\left(\frac{85}{2^3} + 2^2 5^3\frac{J^2}{D} + 49\frac{5^4}{2^3}\frac{J^4}{D^2} + \frac{5^5}{2^2}\frac{J^6}{D^3} - 3.2^{11}5^4\frac{JL}{D^2} - 2^{10}5^5\frac{J^3L}{D^3} + 2^{20}5^5\frac{L^2}{D^3}\right)\Lambda^2$$
$$+\left(\frac{55}{2^4} - 41\frac{5^3}{2^2}\frac{J^2}{D} + 307\frac{5^4}{2^4}\frac{J^4}{D^2} + \frac{5^7}{2^3}\frac{J^6}{D^3} + 9.2^8 5^4\frac{JL}{D^2} - 2^8 5^7\frac{J^3L}{D^3}\right)\Lambda$$
$$-\left(\frac{3}{2^2} + \frac{5^3}{2^2}\frac{J^2}{D}\right)^3 = 0.\dagger$$

On a Method for Reducing the Differential Expression

$$dt \div \sqrt{t-\alpha\,.\,t-\beta\,.\,t-\gamma\,.\,t-\delta},$$

to the Standard Form. By John Griffiths, M.A.

[*Read Feb. 8th*, 1883.]

1. The method of reduction in question, which I propose here to notice briefly, arises from considering the ordinary equation

$$\cos\theta\cos\phi - \Delta\mu\sin\theta\sin\phi = \cos\mu$$

as being equivalent to two others, viz.,

$$t = a + b\cos\mu + b\Delta\mu\sin\theta\sin\phi \quad \text{and} \quad t = a + b\cos\theta\cos\phi,$$

---

* It will be noticed that it is stated in Salmon's *Higher Algebra*, Art. 241, that the coefficient of JD inside the brackets is $\frac{4}{3}$, but the expression there given appears to have been taken from Professor Sylvester's *Trilogy*, in the "Philosophical Transactions of the Royal Society" for 1864, p. 651, where the expression denoted by D is one-fourth of what is taken as D in this paper and in Salmon's *Higher Algebra*.

On the other hand, Professor Sylvester's L is one-fourth of what is here denoted by L, so that Professor Sylvester's formula for Z, which corresponds to C here, should be a multiple of $2^{13}L - J^3 + \frac{4}{3}JD$, as it would appear but for a misprint on line 7 of the previous page, where a 2^9 is printed as 2^7; and the error is not detected by the method of verification employed.

† The calculations in this paper would have been abbreviated by employing, instead of a, b, the expressions used by Professor Sylvester, viz., $p = (a^2 - b^2)^2$ and $q = a^2b^2$.

or, say, $$t = a+b\ \mathrm{cn}\ u_0+b\ \mathrm{dn}\ u_0\ \mathrm{sn}\ u\ \mathrm{sn}\ (u_0-u),$$

and $$t = a+b\ \mathrm{cn}\ u\ \mathrm{cn}\ (u_0-u),$$

if we adopt elliptic-function notation.

2. In the latter equations we have, taking $u_0 = \int_0^{\mu} \frac{d\mu}{\sqrt{1-k^2\sin^2\mu}}$, four disposable constants, a, b, sn u_0, k, and the success of the method consists in showing that $\frac{dt}{\sqrt{t-\alpha\,.\,t-\beta\,.\,t-\gamma\,.\,t-\delta}}$ is equal to Mdu, where α, β, γ, δ are independent quantities and M is a certain multiplier.

Now if $$t = a+b\ \mathrm{cn}\ u_0+b\ \mathrm{dn}\ u_0\ \mathrm{sn}\ u\ \mathrm{sn}\ (u_0-u),$$

and $$t = a+b\ \mathrm{cn}\ u\ \mathrm{cn}\ (u_0-u),$$

we have, by differentiation,

$$\frac{dt}{du} = b\ \mathrm{dn}\ u_0\{\mathrm{cn}\ u\ \mathrm{dn}\ u\ \mathrm{sn}\ (u_0-u)-\mathrm{sn}\ u\ \mathrm{cn}\ (u_0-u)\ \mathrm{dn}\ (u_0-u)\},$$

and $$\frac{dt}{du} = b\ \{\mathrm{cn}\ u\ \mathrm{sn}\ (u_0-u)\ \mathrm{dn}\ (u_0-u)-\mathrm{cn}\ (u_0-u)\ \mathrm{sn}\ u\ \mathrm{dn}\ u\};$$

whence

$$(1-\mathrm{dn}^2 u_0)\left(\frac{dt}{du}\right)^2 = b^2\ \mathrm{dn}^2 u_0\ \{\mathrm{cn}^2 u\ \mathrm{sn}^2\ (u_0-u)-\mathrm{cn}^2\ (u_0-u)\ \mathrm{sn}^2 u\} \times \{\mathrm{dn}^2 u-\mathrm{dn}^2\ (u_0-u)\}$$

$$= b^2\mathrm{dn}^2 u_0\{\mathrm{sn}^2\ (u_0-u)-\mathrm{sn}^2 u\}\,.\,k^2\{\mathrm{sn}^2\ (u_0-u)-\mathrm{sn}^2 u\},$$

i.e., $$\frac{dt}{du} = b\,\frac{\mathrm{dn}\ u_0}{\mathrm{sn}\ u_0}\{\mathrm{sn}^2\ (u_0-u)-\mathrm{sn}^2 u\}.$$

Again, to prove that $\mathrm{sn}^2\ (u_0-u)-\mathrm{sn}^2 u$ is a quartic function of t, of the required form, we may write sn $u = x$ and sn $(u_0-u) = y$, so that $t = a+b\ \mathrm{cn}\ u_0+b\ \mathrm{dn}\ u_0\,xy$ and $t = a+b\sqrt{1-x^2\,.\,1-y^2}$; it is then found, after making some easy reductions, that

$$b^4\ \mathrm{dn}^4 u_0\,(x^2-y^2)^2 = T_1\,.\,T_2,$$

where $$T_1 = (t-a-b\ \mathrm{cn}\ u_0-b\ \mathrm{dn}\ u_0)^2-(t-a)^2\ \mathrm{dn}^2 u_0,$$

$$T_2 = (t-a-b\ \mathrm{cn}\ u_0+b\ \mathrm{dn}\ u_0)^2-(t-a)^2\ \mathrm{dn}^2 u_0,$$

or $T_1 = (1-\mathrm{dn}^2 u_0)(t-a)(t-\beta)$ and $T_2 = (1-\mathrm{dn}^2 u_0)(t-\gamma)(t-\delta)$,

if $$\left.\begin{aligned} a+\left(\frac{\mathrm{cn}\ u_0+\mathrm{dn}\ u_0}{1+\mathrm{dn}\ u_0}\right)b = a, \quad a+\left(\frac{\mathrm{cn}\ u_0+\mathrm{dn}\ u_0}{1-\mathrm{dn}\ u_0}\right)b = \beta \\ a+\left(\frac{\mathrm{cn}\ u_0-\mathrm{dn}\ u_0}{1+\mathrm{dn}\ u_0}\right)b = \gamma, \quad a+\left(\frac{\mathrm{cn}\ u_0-\mathrm{dn}\ u_0}{1-\mathrm{dn}\ u_0}\right)b = \delta \end{aligned}\right\} \ldots\ldots(\mathrm{A}).$$

Hence, ultimately,

$$\frac{dt}{du} = \frac{k^2\ \mathrm{sn}\ u_0}{b\ \mathrm{dn}\ u_0}\sqrt{t-\alpha\,.\,t-\beta\,.\,t-\gamma\,.\,t-\delta} \quad \ldots\ldots\ldots\ldots(\mathrm{B}).$$

3. To complete the transformation, it remains to find from the system (A) the values of a, b, k, and $\operatorname{sn} u_0$ in terms of α, β, γ, δ, which are supposed to be given independent quantities.

The results are

$$a = \frac{\alpha\delta-\beta\gamma}{\alpha-\beta-\gamma+\delta}; \quad -b = \frac{\alpha-\gamma\,.\,\beta-\delta}{\alpha-\beta-\gamma+\delta}; \quad -\operatorname{dn} u_0 = \frac{\alpha-\beta-\gamma+\delta}{\alpha+\beta-\gamma-\delta};$$

$$\operatorname{sn} u_0 = 2\,\frac{\sqrt{\alpha-\delta\,.\,\beta-\gamma}}{\alpha+\beta-\gamma-\delta}; \quad k^2 = \frac{\alpha-\gamma\,.\,\beta-\delta}{\alpha-\delta\,.\,\beta-\gamma}; \quad \operatorname{cn} u_0 = \frac{-\alpha+\beta-\gamma+\delta}{\alpha+\beta-\gamma-\delta};$$

so that (B) may be written in the form

$$dt \div \sqrt{t-\alpha\,.\,t-\beta\,.\,t-\gamma\,.\,t-\delta} = \frac{2}{M}\,du, \text{ where } M = \sqrt{\alpha-\delta\,.\,\beta-\gamma},$$

$$u = \int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \text{ and } t = \frac{\alpha\delta-\beta\gamma}{\alpha-\beta-\gamma+\delta} - \frac{\alpha-\gamma\,.\,\beta-\delta}{\alpha-\beta-\gamma+\delta}\operatorname{cn} u \operatorname{cn}(u_0-u),$$

$$-\operatorname{dn} u_0 = \frac{\alpha-\beta-\gamma+\delta}{\alpha+\beta-\gamma-\delta}, \qquad k = \sqrt{\frac{\alpha-\gamma\,.\,\beta-\delta}{\alpha-\delta\,.\,\beta-\gamma}}.$$

4. As a particular case of the above, we may deduce Landen's transformation as follows:

Let $a=0$, $b=1$, $u_0=K$; so that $\operatorname{cn} u_0 = 0$, $\operatorname{sn} u_0 = 1$, $\operatorname{dn} u_0 = k'$,

$$\alpha = \frac{k'}{1+k'} = -\gamma; \quad \beta = \frac{k'}{1-k'} = -\delta, \text{ and } t = k'\operatorname{sn} u \operatorname{sn}(K-u),$$

where $$k^2+k'^2 = 1.$$

Hence

$$\frac{dt}{\sqrt{a^2-t^2\,.\,\beta^2-t^2}} = \frac{2}{\alpha+\beta}\,du, \quad \text{or} \quad \frac{d\psi}{\sqrt{1-\lambda^2\sin^2\psi}} = (1+k')\,\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

if $$\frac{k'\sin\psi}{1+k'} = t = k'\operatorname{sn} u \operatorname{sn}(K-u), \qquad \lambda = \frac{1-k'}{1+k'},$$

and $$u = \int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

5. If we take the expression $\frac{dt}{\sqrt{T}}$, where T is a quartic function of t of the form $T = (1, p_1, p_2, p_3, p_4)\,(t, 1)^4$, it appears from the foregoing results that it can be reduced to the form $\frac{M dx}{\sqrt{1-x^2\,.\,1-qx^2}}$, where q is given, in terms of the invariants I, J of the quartic, by the equation

$$\frac{(q^2-q+1)^3}{(2-q)^2\,(1-2q)^2\,(1+q)^2} = \frac{4^2}{12^3}\,\frac{I^3}{J^2}.$$

Note on the Mechanical Description of Bicircular Quartics.

By Mr. Harry Hart.

[*Read Feb. 8th*, 1883.]

1. Let Fig. (1) represent the four-piece linkage known as the Sylvester-Kempe Quadruplane, in which

$$AB = CD = a, \qquad AD = BC = b,$$
$$AP = CR = pa, \qquad AS = CQ = pb,$$
$$BP = DR = qa, \qquad DC = BQ = qb,$$

the angles APB, BQC, CRD, DSA each being $= \alpha$.

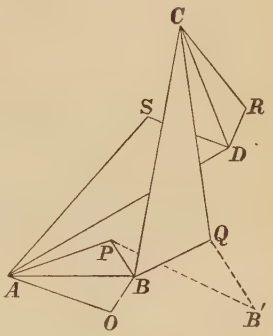


Fig. 1.

Then $PQRS$ is a parallelogram, and

$$PQ \,.\, QR = pq\,(a^2 \sim b^2) = \mu^2,$$

and $$\angle PQR = \alpha \text{ or } \pi - \alpha,$$

according as b is $>$ or $<$ a.

Make the triangle AOB equal to APB in all respects; produce OA, OB making OAF_1, OBF_2 equal to $2OA$, $2OB$ respectively. Then, APB being fixed, the locus of R is the pedal of a conic (foci F_1, F_2, transverse axis $= 2AD = 2b$), O being the pole. Or, the locus is the nodal bicircular quartic which is the envelope of circles through O having their centres on the focal conic (foci A, B, transverse axis $= AD = b$).

The above is Mr. S. Roberts's well-known method of describing the pedal of a central conic; I shall show how it may be extended to the case in which the conic is a parabola.

Let a second quadruplane $A'B'C'D'$ be constructed ($A' \ldots a' \ldots p' \ldots$ corresponding to $A \ldots a \ldots p \ldots$). Then the two quadruplanes can be linked together at P, Q, R, S, provided

$$p'q'\,(a'^2 \sim b'^2) = pq\,(a^2 \sim b^2) = \mu^2,$$

and $\angle A'PB' = \angle APB$ or its supplement, according as $a^2 - b^2$ and $a'^2 - b'^2$ are of the same or opposite sign.

Fix the piece $A'PB'$, and construct the points O', F_1', F_2' corresponding to O, F_1, F_2; then, in the first quadruplane, the piece APB moves about the fixed point P, the point Q describes a circle whose centre is B' and radius $= B'Q$, and the point R describes the pedal of a conic (foci F_1', F_2', tranverse axis $= 2b'$), O' being the pole.

Hence, if the point P of the first quadruplane be fixed, and Q be constrained to describe a circle by means of a bar $B'Q$ $(= r)$ turning about a fixed centre B' $(B'P = c)$, the point R will describe the pedal

of a conic. The position and magnitude of this conic are easily found by means of the auxiliary quadruplane.

For $$q'a' = PB' = c,$$

$$q'b' = QB' = r,$$

and $$\mu^2 = p'q'\,(a'^2 \sim b'^2),$$

$$= \frac{p'}{q'}\,(c^2 \sim r^2),$$

whence $$PA' = p'a' = \frac{\mu^2 c}{c^2 \sim r^2}.$$

Make the angle $B'PA' =$ constant angle PQR (α) or its supplement according as r is $>$ or $<$ c, and set off $PA' = \frac{\mu^2 c}{c^2 \sim r^2}$; also make $A'O$, $B'O = A'P$, $B'P$ respectively, and $O'A'F_1' = 2O'A'$, $O'B'F_2' = 2O'B'$. Then F_1', F_2' are the foci of the conic, and its transverse axis

$$= 2b' = \frac{2r}{c^2 \sim r^2} \cdot \sqrt{\mu^4 - 2\mu^2\,(r^2 - c^2)\cos\alpha + (r^2 - c^2)^2};$$

the conic being an ellipse or hyperbola according as r is $>$ or $<$ c.

I shall now give a separate investigation of the case when $r = c$.

2. In Fig. 2, let P, Q, R, S be the four points, as above, of the quadruplane, AQ the link that constrains Q to describe the circle PQB. In this case the inverse circle described by S becomes the straight line SX, and if XPH be at right angles to SX, $B\widehat{P}H = P\widehat{Q}R = \alpha$. Let the circle PQB meet QR, PA, PH in O, B, H respectively. It is easily seen that $POBH$ is a rectangle, and that consequently QR always passes through a fixed point O.

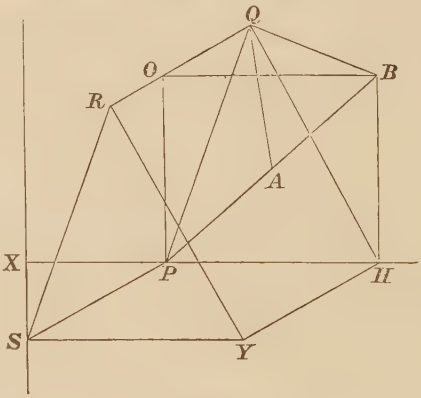


Fig 2.

Complete the parallelogram $PSYH$; then $HQRY$ is a rectangle.

Since $SY = PH =$ const., the locus of Y is a straight line; OR is the perpendicular from O on RY, which is a tangent to the parabola whose focus is H, the locus of Y being the tangent at the vertex. As Q approaches P, R and S approach at infinity, so that, whilst R describes the circular cubic, S is describing its asymptote.

3. Since O is a fixed point, the triangle OAP is invariable, so that, if to the original quadruplane we add the bar AQ and the triangular piece OAP, we have a six-piece linkage having three collinear points R, O, Q.

Let $QR = r$, $OQ = \rho$, $OP = a$; then from the triangle POQ, since

$$PQ = \frac{\mu^2}{r},$$

$$a^2 = \rho^2 - 2\rho \frac{\mu^2}{r} \cos \alpha + \frac{\mu^4}{r^2},$$

so that, fixing the point Q, if O describe the curve

$$\rho = f(\theta),$$

the equation of the locus of R will be

$$a^2 = \{f(\theta)\}^2 - 2\frac{\mu^2}{r} \cos \alpha f(\theta) + \frac{\mu^4}{r^2}.$$

As a particular case, let O describe a circle passing through Q by means of a seventh link $CQ = b = CO$, C being fixed.

Then $\rho = 2b \cos \theta$, and equation of curve described by R is

$$a^2 = 4b^2 \cos^2 \theta - 4b \frac{\mu^2}{r} \cos \alpha \cos \theta + \frac{\mu^4}{r^2},$$

or, transforming to Cartesian coordinates,

$$a^2 (x^2 + y^2) = 4b^2 x^2 - 4b\mu^2 x \cos \alpha + \mu^4,$$

that is, the locus of R is a conic; it will be an ellipse, parabola, or hyperbola, according as a is $>$, $=$, or $< 2b$.

4. It remains to show how any bicircular quartic or circular cubic can be described by linkwork.

The construction given by Dr. Casey for generating any of these curves is as follows:—Upon any tangent to one of the focal conics F drop a perpendicular OY from the centre of the circle J corresponding to that conic; on this perpendicular take two points P, P' such that $PY = P'Y$ and $OP \cdot OP' =$ square of the radius of the circle; then P, P' are points on the quartic.

First, suppose the conic an ellipse or hyperbola.

Let S, S', Fig 3, be the foci of the conic F, O the centre of the centre J.

Construct Mr. S. Roberts's 3-bar linkwork, making the bars SpR', $S'pR$ each equal to the transverse axis of F, and the triangular piece RQR' equal in all respects to SOS'. Connect O, Q with the two extreme poles of a reciprocator whose four collinear points O, P, P', Q satisfy the relation

$$OP \cdot OP' = OP \cdot PQ = \&c.$$
$$= \text{square of the radius of } J.$$

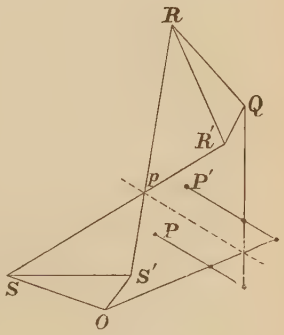


Fig. 3.

Then evidently the line bisecting PP' at right angles is a tangent to the conic F at p, and consequently P, P' describe the required quartic.

So, if the conic be a parabola, if the poles of the reciprocator be attached to the points O, R of the linkwork represented in Fig. 2, the points P, P' will trace out the corresponding circular cubic.

[See a paper by Prof. Cayley on the "Mechanical Description of a Cartesian," *Quarterly Mathematical Journal*, Vol. viii. 328–330.]

Thursday, May 10th, 1883.

SAMUEL ROBERTS, Esq., F.R.S., Vice-President, in the Chair.

Mr. M. J. M. Hill, M.A., Professor of Mathematics in the Mason Science College, Birmingham, was elected a member.

Professor Cayley made a communication "On Mr. Wilkinson's Rectangular Transformation." Mr. Tucker read an abstract of a paper by Professor Genese, entitled "Relations between the Common Points and Common Tangents of Two Conics"; a paper on "Two Concentric Circles," by himself; and abstracts of papers "On the Motion of a Particle on the Surface of an Ellipsoid," by Mr. W. R. W. Roberts; and "Unicursal Twisted Quartics, Part II.," by Mr. R. A. Roberts. The Rev. M. M. U. Wilkinson communicated a second paper on "Spherical Functions."

The following presents were received :—

"Annali di Matematica pura ed applicata," Serie 11ª, Tomo xi., Fasc. 4° (Aprile, 1883); Milano.

"American Journal of Mathematics," Vol. v., No. 3.

"Proceedings of the Royal Society," Vol. xxxv., No. 224.

"Educational Times," May, 1883.

"U. S. Coast and Geodetic Survey," A Treatise on Projections, by Thomas Craig; 4to, Washington, 1882.

"Crelle; Journal für die reine und angewandte Mathematik," Bd. xciv., Heft 2.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," 1882, Parts 39 to 54 and Index.

"Beiblätter zu den Annalen der Physik und Chemie," Band vii., Stück 3, 1883, No. 3.

"Archives Néerlandaises des Sciences exactes et naturelles," T. xvii., Parts 3, 4, and 5; T. xviii., Part 1; Harlem.

"Bulletin des Sciences Mathématiques," 2° Serie, T. vii., February, 1883.

"Atti della R. Accademia dei Lincei, &c.," Vol. vii., Fasc. 7 and 8; Roma.

"Bulletin Astronomique et Météorologique de l'Observatoire Impérial de Rio de Janeiro," January and February, 1883.

On Elliptic Function Formulæ connected with the Transformation of Rectangular Coordinates.

By Rev. M. M. U. Wilkinson.

[*Read April 12th*, 1883, *and May 10th*, 1883.]

1. We define the nine quantities

$$\beta,\ \varpi,\ \phi,\ \gamma,\ \kappa,\ \chi,\ \delta,\ \tau,\ \theta$$

by the following equations:

$$\left.\begin{aligned}
\beta^2 &= k'^2 \operatorname{cn}(q-t)\operatorname{dn}(q-t) &\div\\
\varpi^2 &= k^2 \operatorname{cn}^2 p \operatorname{dn}(q-t)\operatorname{cn}(t-p)\operatorname{cn}(p-q) &\div\\
\phi^2 &= -\operatorname{dn}^2 p \operatorname{cn}(q-t)\operatorname{dn}(t-p)\operatorname{dn}(p-q) &\div\\
\text{denominator} &= k^2k'^2 \operatorname{sn}(t-p)\operatorname{sn}(p-q)\{\operatorname{sn}^2(q-t)-\operatorname{sn}^2 p\}
\end{aligned}\right\}\ldots(1).$$

$$\left.\begin{aligned}
\gamma^2 &= k'^2 \operatorname{cn}(t-p)\operatorname{dn}(t-p) &\div\\
\kappa^2 &= k^2 \operatorname{cn}^2 q \operatorname{dn}(t-p)\operatorname{cn}(p-q)\operatorname{cn}(q-t) &\div\\
\chi^2 &= -\operatorname{dn}^2 q \operatorname{cn}(t-p)\operatorname{dn}(p-q)\operatorname{dn}(q-t) &\div\\
\text{denominator} &= k^2k'^2 \operatorname{sn}(p-q)\operatorname{sn}(q-t)\{\operatorname{sn}^2(t-p)-\operatorname{sn}^2 q\}
\end{aligned}\right\}\ldots(2).$$

$$\left.\begin{aligned}
\delta^2 &= k'^2 \operatorname{cn}(p-q)\operatorname{dn}(p-q) &\div\\
\tau^2 &= k^2 \operatorname{cn}^2 t \operatorname{dn}(p-q)\operatorname{cn}(q-t)\operatorname{cn}(t-p) &\div\\
\theta^2 &= -\operatorname{dn}^2 t \operatorname{cn}(p-q)\operatorname{dn}(q-t)\operatorname{dn}(t-p) &\div\\
\text{denominator} &= k^2k'^2 \operatorname{sn}(q-t)\operatorname{sn}(t-p)\{\operatorname{sin}^2(p-q)-\operatorname{sn}^2 t\}
\end{aligned}\right\}\ldots(3).$$

When writing at full length, we shall write $\beta\,(p, q, t, k)$ for β, and so for the others. Thus we have

$$\left.\begin{aligned}
\gamma\,(p, q, t, k) &= \beta\,(q, t, p, k)\\
\delta\,(p, q, t, k) &= \beta\,(t, p, q, k)\\
\kappa\,(p, q, t, k) &= \varpi\,(q, t, p, k)\\
\tau\,(p, q, t, k) &= \varpi\,(t, p, q, k)\\
\chi\,(p, q, t, k) &= \phi\,(q, t, p, k)\\
\theta\,(p, q, t, k) &= \phi\,(t, p, q, k)
\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The cyclical character of these formulæ is obvious. It will be convenient to express them all (as γ, δ are here done) as β functions. We shall show that this can be done. We may call β, ϖ, ϕ, p labials, γ, κ, χ, q gutturals, and δ, τ, θ, t dentals; also we may call β, γ, δ smooths, ϖ, κ, τ roughs, and ϕ, χ, θ aspirates. And by arguments we mean p, q, t; by functions, β, γ, &c.

We may notice that β, &c. can be expressed factorially by means of Θ functions.

2. Since

$$k'^2 \operatorname{cn}(q-t)\operatorname{dn}(q-t) = (\operatorname{dn}^2 p - k^2 \operatorname{cn}^2 p)\operatorname{cn}(q-t)\operatorname{dn}(q-t),$$

$$\begin{aligned}
k^2 \operatorname{cn}^2 p \operatorname{dn}(q-t)\operatorname{cn}(t-p)\operatorname{cn}(p-q) &= k^2 \operatorname{cn}^2 p \operatorname{cn}(q-t)\operatorname{dn}(q-t)\\
&\quad + k^2 \operatorname{sn}(t-p)\operatorname{sn}(p-q)\operatorname{cn}^2 p \operatorname{dn}^2(q-t)\\
-\operatorname{dn}^2 p \operatorname{cn}(q-t)\operatorname{dn}(t-p)\operatorname{dn}(p-q) &= -\operatorname{dn}^2 p \operatorname{cn}(q-t)\operatorname{dn}(q-t)\\
&\quad - k^2 \operatorname{sn}(t-p)\operatorname{sn}(p-q)\operatorname{dn}^2 p \operatorname{cn}^2(q-t),
\end{aligned}$$

$$\operatorname{cn}^2 p \operatorname{dn}^2(q-t) - \operatorname{dn}^2 p \operatorname{cn}^2(q-t) = k'^2\{\operatorname{sn}^2(q-t) - \operatorname{sn}^2 p\},$$

we have,

$$\left.\begin{aligned}\beta^2+\varpi^2+\phi^2&=1\\ \gamma^2+\kappa^2+\chi^2&=1\\ \delta^2+\tau^2+\theta^2&=1\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

3. Our definitions are as yet ambiguous as to the signs of β, &c. But we have

$$\frac{\varpi^2\kappa^2}{\beta^2\gamma^2}=\frac{k^4\,\mathrm{cn}^2p\,\mathrm{cn}^2q\,\mathrm{cn}^2(p-q)}{k'^4},$$

$$\frac{\phi^2\chi^2}{\beta^2\gamma^2}=\frac{\mathrm{dn}^2p\,\mathrm{dn}^2q\,\mathrm{dn}^2(p-q)}{k'^4}.$$

We shall always take

$$\left.\begin{aligned}\frac{\varpi\kappa}{\beta\gamma}&=\frac{k^2\,\mathrm{cn}\,p\,\mathrm{cn}\,q\,\mathrm{cn}\,(p-q)}{k'^2}\\ \frac{\phi\chi}{\beta\gamma}&=-\frac{\mathrm{dn}\,p\,\mathrm{dn}\,q\,\mathrm{dn}\,(p-q)}{k'^2}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

And similarly for the others. Thus we have

$$\left.\begin{aligned}\beta\gamma+\varpi\kappa+\phi\chi&=0\\ \gamma\delta+\kappa\tau+\chi\theta&=0\\ \delta\beta+\tau\varpi+\theta\phi&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Equations (5) and (6) show that our elliptic function formulæ express the nine quantities which are used in the Transformation of Rectangular Coordinates.

4. A system of β functions will be called positive or negative according as $+1$ or -1 is the value of the determinant

$$\begin{vmatrix}\beta,&\gamma,&\delta\\ \varpi,&\kappa,&\tau\\ \phi,&\chi,&\theta\end{vmatrix}.$$

When nothing is said to the contrary, it will be assumed that the system is a positive system.

5. The β functions lead to several interesting formulæ. We will give some of the simplest.

$$\left.\begin{aligned}\frac{\varpi\chi}{\phi\kappa}&=\frac{\mathrm{cn}\,p\,\mathrm{dn}\,q\,\mathrm{cn}\,(t-p)\,\mathrm{dn}\,(q-t)}{\mathrm{dn}\,p\,\mathrm{cn}\,q\,\mathrm{cn}\,(q-t)\,\mathrm{dn}\,(t-p)}\\ \frac{\phi\gamma}{\beta\chi}&=\frac{\mathrm{dn}\,p\,\mathrm{dn}\,(t-p)}{\mathrm{dn}\,q\,\mathrm{dn}\,(q-t)}\\ \frac{\beta\kappa}{\varpi\gamma}&=\frac{\mathrm{cn}\,q\,\mathrm{cn}\,(q-t)}{\mathrm{cn}\,p\,\mathrm{cn}\,(t-p)}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Since we have

$$\mathrm{dn}\,q\,\mathrm{cn}\,t\,\mathrm{cn}\,(t-p)\,\mathrm{dn}\,(p-q)-\mathrm{cn}\,q\,\mathrm{dn}\,t\,\mathrm{cn}\,(p-q)\,\mathrm{dn}\,(t-p)$$
$$=k'^2\{\mathrm{sn}\,t\,\mathrm{sn}\,(t-p)+\mathrm{sn}\,q\,\mathrm{sn}\,(p-q)\},$$

$$\operatorname{cn} t \operatorname{cn}(t-p) - \operatorname{cn} q \operatorname{cn}(p-q) = -\operatorname{dn} p \{\operatorname{sn} t \operatorname{sn}(t-p) + \operatorname{sn} q \operatorname{sn}(p-q)\},$$
$$\operatorname{dn} t \operatorname{dn}(t-p) - \operatorname{dn} q \operatorname{dn}(p-q) = -k^2 \operatorname{cn} p \{\operatorname{sn} t \operatorname{sn}(t-p) + \operatorname{sn} q \operatorname{sn}(p-q)\},$$

we may write

$$\left.\begin{aligned} \frac{\beta}{\chi\tau} &= \frac{k'^2\{\operatorname{sn} q \operatorname{sn}(p-q) + \operatorname{sn} t \operatorname{sn}(t-p)\}}{\operatorname{dn} q \operatorname{cn} t \operatorname{dn}(p-q) \operatorname{cn}(t-p)} \\ \frac{\varpi}{\gamma\theta} &= \frac{k^2 \operatorname{cn} p \{\operatorname{sn} q \operatorname{sn}(p-q) + \operatorname{sn} t \operatorname{sn}(t-p)\}}{\operatorname{dn} t \operatorname{dn}(t-p)} \\ \frac{\phi}{\kappa\delta} &= \frac{-\operatorname{dn} p \{\operatorname{sn} q \operatorname{sn}(p-q) + \operatorname{sn} t \operatorname{sn}(t-p)\}}{\operatorname{cn} q \operatorname{cn}(p-q)} \end{aligned}\right\} \quad \ldots\ldots\ldots(9).$$

Let

$$Q = \{\operatorname{sn} q \operatorname{sn}(p-q) + \operatorname{sn} t \operatorname{sn}(t-p)\}\{\operatorname{sn} t \operatorname{sn}(q-t) + \operatorname{sn} p \operatorname{sn}(p-q)\} \times \{\operatorname{sn} p \operatorname{sn}(t-p) + \operatorname{sn} q \operatorname{sn}(q-t)\} \ldots\ldots\ldots\ldots\ldots(10).$$

Then shall

$$\left.\begin{aligned} \beta\chi\tau &= -\operatorname{cn} t \operatorname{dn} q \operatorname{cn}(t-p) \operatorname{cn}(q-t) \operatorname{dn}(p-q) \operatorname{dn}(q-t) \div \\ \beta\kappa\theta &= -\operatorname{cn} q \operatorname{dn} t \operatorname{cn}(p-q) \operatorname{cn}(q-t) \operatorname{dn}(t-p) \operatorname{dn}(q-t) \div \\ \text{denominator} &= k^2 k'^2 Q \end{aligned}\right\} \ldots(11).$$

$$Q^2 = \{\operatorname{sn}^2 p - \operatorname{sn}^2(q-t)\}\{\operatorname{sn}^2 q - \operatorname{sn}^2(t-p)\}\{\operatorname{sn}^2 t - \operatorname{sn}^2(p-q)\} \times \operatorname{sn}^2(q-t) \operatorname{sn}^2(t-p) \operatorname{sn}^2(p-q) \ldots\ldots\ldots\ldots\ldots(12).$$

It is obvious that three more formulæ, of almost equal interest, can be obtained by increasing p, q, t, in Q, Q^2, by K, iK', and $K+iK'$.

6. In equations (4) we express, by a cyclical change of the arguments, guttural and dental functions by labial functions. We shall now show how to express rough and aspirate functions by smooth functions. We shall thus be able to express any of the nine functions in a set by a β function.

The change of k into $\frac{1}{k'}$, $\frac{1}{k'}$ into $-\frac{ik'}{k}$, and $-\frac{ik'}{k}$ into k, is a cyclical change. In order to change the modulus from k into $\frac{1}{k'}$ or from k into $-\frac{ik'}{k}$, we require the formulæ

$$\left.\begin{aligned} \operatorname{sn}(u, k) &= -i \operatorname{sn}\left(ik'u, \frac{1}{k'}\right) \div \\ \operatorname{cn}(u, k) &= k' \qquad\qquad\qquad \div \\ \operatorname{dn}(u, k) &= k' \operatorname{cn}\left(ik'u, \frac{1}{k'}\right) \div \\ \text{where denominator} &= k' \operatorname{dn}\left(ik'u, \frac{1}{k'}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots(13),$$

$$\left.\begin{aligned} \operatorname{sn}(u,k) &= -i \operatorname{sn}\left(iku, \frac{-ik'}{k}\right) \div \\ \operatorname{cn}(u,k) &= k \operatorname{dn}\left(iku, -\frac{ik'}{k}\right) \div \\ \operatorname{dn}(u,k) &= k \div \\ \text{where denominator} &= k \operatorname{cn}\left(iku, -\frac{ik'}{k}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots(14).$$

When the modulus is sufficiently indicated by the coefficient of the argument, we shall not think it necessary always to express it.

7. In the two expressions,

$$\beta(p+K,\ q+K,\ t+K,\ k),$$
$$\beta\left(p+K,\ q+K,\ t+K,\ \frac{1}{k'}\right),$$

K has a different meaning, meaning $K(k)$ in the first, and $K\left(\frac{1}{k'}\right)$ in the second.

The equations which we shall want for transforming K, K' from one modulus to any of the other five moduli connected with it, are, to a sign *près*,

$$\left.\begin{aligned} K(k) = K'(k') &= \frac{1}{k'} K'\left(\frac{1}{k'}\right) = \frac{1}{k'} K\left(\frac{ik}{k'}\right) \\ &= \frac{1}{k} K\left(\frac{1}{k}\right) + \frac{i}{k} K'\left(\frac{1}{k}\right) = \frac{1}{k} K'\left(\frac{ik'}{k}\right) + \frac{i}{k} K\left(\frac{ik'}{k}\right) \\ K'(k) = K(k') &= \frac{1}{k} K'\left(\frac{1}{k}\right) = \frac{1}{k} K\left(\frac{ik'}{k}\right) \\ &= \frac{1}{k'} K\left(\frac{1}{k'}\right) + \frac{i}{k'} K'\left(\frac{1}{k'}\right) = \frac{1}{k'} K'\left(\frac{ik}{k'}\right) + \frac{i}{k'} K\left(\frac{ik}{k'}\right) \\ K\left(\frac{1}{k'}\right) = K'\left(\frac{ik}{k'}\right) &= k'K(k') + ik'K'(k') = \frac{ik'}{k} K'\left(\frac{ik'}{k}\right) \\ &= \frac{ik'}{k} K\left(\frac{1}{k}\right) = k'K'(k) + ik'K(k) \\ K'\left(\frac{1}{k'}\right) = k'K(k) &= k'K'(k') = \frac{k'}{k} K\left(\frac{1}{k}\right) + \frac{ik'}{k} K'\left(\frac{1}{k}\right) \\ &= K\left(\frac{ik}{k'}\right) = \frac{k'}{k} K\left(\frac{ik'}{k}\right) + \frac{ik'}{k} K\left(\frac{ik'}{k}\right) \end{aligned}\right\} \quad \ldots(15),$$

and so on: consistency of sign in these equations is not, at present, of any importance.

8. We have then, the modulus in every case being sufficiently indicated by the coefficient of the argument,

$$k^2k'^2 \operatorname{sn}(t-p) \operatorname{sn}(p-q)\{\operatorname{sn}^2(q-t) - \operatorname{sn}^2 p\}$$
$$= \frac{k^2 \operatorname{sn} ik'(t-p) \operatorname{sn} ik'(p-q)\{\operatorname{sn}^2 ik'(q-t) - \operatorname{sn}^2 ik'p\}}{k'^2 \operatorname{dn} ik'(t-p) \operatorname{dn} ik'(p-q) \operatorname{dn}^2 ik'p \operatorname{dn}^2 ik'(q-t)};$$

$$k^2 \operatorname{cn}^2 p \operatorname{dn}(q-t) \operatorname{cn}(t-p) \operatorname{cn}(p-q)$$
$$= \frac{k^2}{\operatorname{dn}^2 ik'p} \cdot \frac{\operatorname{cn} ik'(q-t)}{\operatorname{dn} ik'(q-t) \operatorname{dn} ik'(t-p) \operatorname{dn} ik'(p-q)};$$

therefore

$$\varpi^2(p, q, t, k) = \beta^2\left(ik'p, ik'q, ik't, \frac{1}{k'}\right) \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Again,

$$k^2k'^2 \operatorname{sn}(t-p) \operatorname{sn}(p-q)\{\operatorname{sn}^2(q-t) - \operatorname{sn}^2 p\}$$
$$= \frac{k'^2 \operatorname{sn} ik(t-p) \operatorname{sn} ik(p-q)\{\operatorname{sn}^2 ik(q-t) - \operatorname{sn}^2 ikp\}}{k^2 \operatorname{cn}^2 ikp \operatorname{cn}^2 ik(q-t) \operatorname{cn} ik(t-p) \operatorname{cn} ik(p-q)},$$

$$-\operatorname{dn}^2 p \operatorname{cn}(q-t) \operatorname{dn}(t-p) \operatorname{dn}(p-q)$$
$$= -\frac{1}{\operatorname{cn}^2 ikp} \cdot \frac{\operatorname{dn} ik(q-t)}{\operatorname{cn} ik(q-t) \operatorname{cn} ik(t-p) \operatorname{cn} ik(p-q)};$$

therefore

$$\phi^2(p, q, t, k) = \beta^2\left(ikp, ikq, ikt, \frac{ik'}{k}\right) \ldots\ldots\ldots\ldots\ldots\ldots (17).$$

9. As we have

$$\beta^2 = -k^2\left(1-\frac{1}{k^2}\right) \operatorname{dn}\left(kq-kt, \frac{1}{k}\right) \operatorname{cn}\left(kq-kt, \frac{1}{k}\right) \div$$

$$\varpi^2 = k^2 \operatorname{dn}^2\left(kp, \frac{1}{k}\right) \operatorname{cn}\left(kq-kt, \frac{1}{k}\right) \operatorname{dn}\left(kt-kp, \frac{1}{k}\right) \operatorname{dn}\left(kp-kq, \frac{1}{k}\right) \div$$

$$\phi^2 = -\operatorname{cn}^2\left(kp, \frac{1}{k}\right) \operatorname{dn}\left(kq-kt, \frac{1}{k}\right) \operatorname{cn}\left(kt-kp, \frac{1}{k}\right) \operatorname{dn}\left(kp-kq, \frac{1}{k}\right) \div$$

$$\text{denominator} = \frac{1-k^2}{k^2} \operatorname{sn}\left(kt-kp, \frac{1}{k}\right) \operatorname{sn}\left(kp-kq, \frac{1}{k}\right)$$
$$\times \left\{\operatorname{sn}^2\left(kq-kt, \frac{1}{k}\right) - \operatorname{sn}^2\left(kp, \frac{1}{k}\right)\right\}.$$

These equations show that

$$\left.\begin{aligned} \beta^2(p, q, t, k) &= \beta^2\left(kp, kq, kt, \frac{1}{k}\right) \\ \varpi^2(p, q, t, k) &= \phi^2\left(kp, kq, kt, \frac{1}{k}\right) \\ \phi^2(p, q, t, k) &= \varpi^2\left(kp, kq, kt, \frac{1}{k}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

10. The sets of formulæ obtained by increasing p, q, t each by K, iK', $K+iK'$, are as follows. We may distinguish them by writing $\beta(K)$, $\beta(iK')$, $\beta(K+iK')$, &c.

$$\left.\begin{aligned} \beta^2(K) &= -\operatorname{dn}^2 p \operatorname{cn}(q-t) \operatorname{dn}(q-t) \quad \div \\ \varpi^2(K) &= -k^2 \operatorname{sn}^2 p \operatorname{dn}(q-t) \operatorname{cn}(t-p) \operatorname{cn}(p-q) \quad \div \\ \phi^2(K) &= \operatorname{cn}(q-t) \operatorname{dn}(t-p) \operatorname{dn}(p-q) \quad \div \\ \text{denominator} &= k^2 \operatorname{sn}(t-p) \operatorname{sn}(p-q)\{\operatorname{cn}^2 p \operatorname{cn}^2(q-t) \\ \text{&c.} \quad \ldots\ldots\ldots\ldots\ldots\

$$\left.\begin{aligned}
\beta^2(iK') &= -k'^2\,\mathrm{sn}^2 p\,\mathrm{cn}\,(q-t)\,\mathrm{dn}\,(q-t) &&\div\\
\varpi^2(iK') &= \mathrm{dn}^2 p\,\mathrm{dn}\,(q-t)\,\mathrm{cn}\,(t-p)\,\mathrm{cn}\,(p-q) &&\div\\
\phi^2(iK') &= -\mathrm{cn}^2 p\,\mathrm{cn}\,(q-t)\,\mathrm{dn}\,(t-p)\,\mathrm{dn}\,(p-q) &&\div\\
\text{denominator} &= k'^2\,\mathrm{sn}\,(t-p)\,\mathrm{sn}\,(p-q)\,\{1-k^2\,\mathrm{sn}^2 p\,\mathrm{sn}^2(q-t)\}\\
&\&c.
\end{aligned}\right\}\quad\ldots(20),$$

$$\left.\begin{aligned}
\beta^2(K+iK') &= -\mathrm{cn}^2 p\,\mathrm{cn}\,(q-t)\,\mathrm{dn}\,(q-t) &&\div\\
\varpi^2(K+iK') &= \mathrm{dn}\,(q-t)\,\mathrm{cn}\,(t-p)\,\mathrm{cn}\,(p-q) &&\div\\
\phi^2(K+iK') &= -\mathrm{sn}^2 p\,\mathrm{cn}\,(q-t)\,\mathrm{dn}\,(t-p)\,\mathrm{dn}\,(p-q) &&\div\\
\text{denominator} &= \mathrm{sn}\,(t-p)\,\mathrm{sn}\,(p-q)\,\{k'^2+k^2\,\mathrm{cn}^2 p\,\mathrm{cn}^2(q-t)\}\\
&\&c.
\end{aligned}\right\}\quad\ldots(21).$$

11. Since there are four quantities p, q, t, k, they cannot be expressed in terms of the nine quantities β, &c. which are connected by the six equations (5) and (7). The question which naturally presents itself is to express them in terms of β, &c. and an arbitrary parameter. We will consider two cases: one, in which k is given; the other, in which another set of functions, β', ϖ', &c. are introduced, and defined by the equations

$$\left.\begin{aligned}
\beta'(p,q,t,k) &= \beta(p+r,\,q+r,\,t+r,\,k)\\
\varpi'(p,q,t,k) &= \varpi(p+r,\,q+r,\,t+r,\,k)\\
&\&c.
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(22).$$

And we shall, for convenience, put

$$\left.\begin{aligned}
\mathrm{cn}\,(p-q)\,\mathrm{cn}\,(q-t)\,\mathrm{cn}\,(t-p) &= m^2\\
\mathrm{dn}\,(p-q)\,\mathrm{dn}\,(q-t)\,\mathrm{dn}\,(t-p) &= n^2
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(23).$$

So that

$$\left.\begin{aligned}
n^2 &= k^2m^2+k'^2\\
k^2(1-m^2) &= 1-n^2\\
k'^2(1-m^2) &= n^2-m^2
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(24).$$

There will be one relation connecting the β', &c. functions with the β, &c. functions; and p, q, t, r, k can be determined in terms of β, γ, ... β', γ', ...

12. *Formation of the equations for m^2, n^2, when k is given.*

We have $\varpi^2k'^2-\beta^2k^2m^2 = \dfrac{\mathrm{dn}\,(q-t)\,\mathrm{cn}\,(t-p)\,\mathrm{cn}\,(p-q)}{\mathrm{sn}\,(t-p)\,\mathrm{sn}\,(p-q)}\quad\ldots\ldots\ldots(25)$;

$$\beta^2k^2m^2+(1-\varpi^2)\,k'^2 = -\frac{\mathrm{cn}\,(q-t)\,\mathrm{dn}\,(t-p)\,\mathrm{dn}\,(p-q)}{\mathrm{sn}\,(t-p)\,\mathrm{sn}\,(p-q)}\quad\ldots\ldots(26).$$

Whence the equation

$$\frac{m^2}{k^2m^2+k'^2} = \frac{(\varpi^2k'^2-\beta^2k^2m^2)(\kappa^2k'^2-\gamma^2k^2m^2)(\tau^2k'^2-\delta^2k^2m^2)}{(\varpi^2k'^2-\beta^2k^2m^2-k'^2)(\kappa^2k'^2-\gamma^2k^2m^2-k'^2)(\tau^2k'^2-\delta^2k^2m^2-k'^2)}$$

$$\ldots\ldots\ldots\ldots\ldots(27),$$

an equation which may be written in the somewhat simpler form

$$\frac{m^2}{1-m^2} = -\frac{(\varpi^2k'^2-\beta^2k^2m^2)(\kappa^2k'^2-\gamma^2k^2m^2)(\tau^2k'^2-\delta^2k^2m^2)}{\left\{\begin{array}{c}(\varpi^2\kappa^2+\kappa^2\tau^2+\tau^2\varpi^2)\,k'^4+(\beta^2\varpi^2+\gamma^2\kappa^2+\delta^2\tau^2)\,k^2k'^2m^2\\+(\beta^2\gamma^2+\gamma^2\delta^2+\delta^2\beta^2)\,k^4m^4\end{array}\right\}} \quad \ldots(28)$$

And, in like manner, we have,

$$\frac{k^2n^2}{n^2-k'^2} = \frac{(\phi^2k'^2+\beta^2n^2)(\chi^2k'^2+\gamma^2n^2)(\theta^2k'^2+\delta^2n^2)}{(\phi^2k'^2+\beta^2n^2-k'^2)(\chi^2k'^2+\gamma^2n^2-k'^2)(\theta^2k'^2+\delta^2n^2-k'^2)} \ldots(29),$$

which we may write,

$$\frac{n^2}{1-n^2} = \frac{(\phi^2k'^2+\beta^2n^2)(\chi^2k'^2+\gamma^2n^2)(\theta^2k'^2+\delta^2n^2)}{\left\{\begin{array}{c}(\phi^2\chi^2+\chi^2\theta^2+\theta^2\phi^2)\,k'^4-(\beta^2\phi^2+\gamma^2\chi^2+\delta^2\theta^2)\,k'^2n^2\\+(\beta^2\gamma^2+\gamma^2\delta^2+\delta^2\beta^2)\,n^4\end{array}\right\}} \ldots(30).$$

13. *To express p, q, t, r, k in terms of the β and β' functions, and to obtain the relation connecting these functions.*

We may obviously take the signs of m and n, so that

$$\left.\begin{aligned}\frac{m}{\mathrm{cn}\,(q-t)} &= \frac{k'}{k\,\mathrm{cn}\,p}\,\frac{\varpi}{\beta} = \frac{k'}{k\,\mathrm{cn}\,(p+r)}\,\frac{\varpi'}{\beta'}\\ \frac{n}{\mathrm{dn}\,(q-t)} &= \frac{ik'}{\mathrm{dn}\,p}\,\frac{\phi}{\beta} = \frac{ik'}{\mathrm{dn}\,(p+r)}\,\frac{\phi'}{\beta'}\end{aligned}\right\} \ldots\ldots\ldots(31).$$

Equation (25) gives

$$\left.\begin{aligned}\frac{k^2m^2}{k'^2} &= \frac{\varpi'^2-\varpi^2}{\beta'^2-\beta^2} = \frac{\kappa'^2-\kappa^2}{\gamma'^2-\gamma^2} = \frac{\tau'^2-\tau^2}{\delta'^2-\delta^2}\\ \text{or}\quad -\frac{n^2}{k'^2} &= \frac{\phi'^2-\phi^2}{\beta'^2-\beta^2} = \frac{\chi'^2-\chi^2}{\gamma'^2-\gamma^2} = \frac{\theta'^2-\theta^2}{\delta'^2-\delta^2}\end{aligned}\right\} \ldots\ldots\ldots(32);$$

expressing the one relation connecting the two sets of functions.

From equation (25) we find

$$\varpi^2k'^2+(1-\beta^2)\,k^2m^2 = \frac{\mathrm{cn}\,(t-p)\,\mathrm{cn}\,(p-q)\,\mathrm{dn}\,(t-p)\,\mathrm{dn}\,(p-q)}{\mathrm{sn}\,(t-p)\,\mathrm{sn}\,(p-q)},$$

$$\begin{aligned}\frac{\mathrm{cn}\,(t-p)\,\mathrm{cn}\,(p-q)}{\mathrm{cn}\,(q-t)} &= -\frac{\varpi^2k'^2+(1-\beta^2)\,k^2m^2}{\phi^2k'^2+\beta^2n^2}\\ &= -\frac{\varpi^2(\beta'^2-\beta^2)+(1-\beta^2)(\varpi'^2-\varpi^2)}{\phi^2(\beta'^2-\beta^2)-\beta^2(\phi'^2-\phi^2)}\\ &= \frac{\varpi^2\phi'^2-\phi^2\varpi'^2}{\phi^2\beta'^2-\beta^2\phi'^2} \quad \ldots\ldots\ldots(33);\end{aligned}$$

and, in like manner,

$$\begin{aligned}\frac{\mathrm{dn}\,(t-p)\,\mathrm{dn}\,(p-q)}{\mathrm{dn}\,(q-t)} &= \frac{\varpi^2(\beta'^2-\beta^2)+(1-\beta^2)(\varpi'^2-\varpi^2)}{\varpi^2(\beta'^2-\beta^2)-\beta^2(\varpi'^2-\varpi^2)}\\ &= \frac{\phi^2\varpi'^2-\varpi^2\phi'^2}{\varpi^2\beta'^2-\beta^2\varpi'^2} \quad \ldots\ldots\ldots(34);\end{aligned}$$

and so we obtain

$$\mathrm{cn}^2(q-t) = \frac{(\kappa^2\chi'^2-\chi^2\kappa'^2)(\tau^2\theta'^2-\theta^2\tau'^2)}{(\chi^2\gamma'^2-\gamma^2\chi'^2)(\theta^2\delta'^2-\delta^2\theta'^2)} \quad \ldots\ldots\ldots\ldots(35),$$

$$\mathrm{dn}^2(q-t) = \frac{(\kappa^2\chi'^2-\chi^2\kappa'^2)(\tau^2\theta'^2-\theta^2\tau'^2)}{(\kappa^2\gamma'^2-\gamma^2\kappa'^2)(\tau^2\delta'^2-\delta^2\tau'^2)} \quad \ldots\ldots\ldots\ldots(36);$$

$$\mathrm{cn}^2 p = \frac{\varpi^2(\beta'^2-\beta^2)(\kappa^2\chi'^2-\chi^2\kappa'^2)(\tau^2\theta'^2-\theta^2\tau'^2)}{\beta^2(\varpi'^2-\varpi^2)(\chi^2\gamma'^2-\gamma^2\chi'^2)(\theta^2\delta'^2-\delta^2\theta'^2)} \quad \ldots\ldots\ldots(37),$$

$$\mathrm{dn}^2 p = \frac{\phi^2(\beta'^2-\beta^2)(\kappa^2\chi'^2-\chi^2\kappa'^2)(\tau^2\theta'^2-\theta^2\tau'^2)}{\beta^2(\phi'^2-\phi^2)(\kappa^2\gamma'^2-\gamma^2\kappa'^2)(\tau^2\delta'^2-\delta^2\tau'^2)} \quad \ldots\ldots\ldots(38);$$

other formulæ are

$$\frac{k^2\,\mathrm{sn}(t-p)\,\mathrm{sn}(p-q)\,\mathrm{cn}(q-t)}{\mathrm{dn}(q-t)} = \frac{\varpi'^2-\varpi^2}{\varpi^2\beta'^2-\beta^2\varpi'^2} \quad \ldots\ldots\ldots(39),$$

$$\frac{\mathrm{sn}(t-p)\,\mathrm{sn}(p-q)\,\mathrm{dn}(q-t)}{\mathrm{cn}(q-t)} = \frac{\phi'^2-\phi^2}{\phi^2\beta'^2-\beta^2\phi'^2} \quad \ldots\ldots\ldots(40),$$

$$k^2 = \frac{(\varpi'^2-\varpi^2)(\phi^2\beta'^2-\beta^2\phi'^2)(\chi^2\gamma'^2-\gamma^2\chi'^2)(\theta^2\delta'^2-\delta^2\theta'^2)}{(\phi'^2-\phi^2)(\varpi^2\beta'^2-\beta^2\varpi'^2)(\kappa^2\gamma'^2-\gamma^2\kappa'^2)(\tau^2\delta'^2-\delta^2\tau'^2)} \quad \ldots\ldots(41),$$

$$m^2 = \frac{(\varpi^2\phi'^2-\phi^2\varpi'^2)(\kappa^2\chi'^2-\chi^2\kappa'^2)(\tau^2\theta'^2-\theta^2\tau'^2)}{(\phi^2\beta'^2-\beta^2\phi'^2)(\chi^2\gamma'^2-\gamma^2\chi'^2)(\theta^2\delta'^2-\delta^2\theta'^2)} \quad \ldots\ldots\ldots\ldots(42),$$

$$n^2 = -\frac{(\varpi^2\phi'^2-\phi^2\varpi'^2)(\kappa^2\chi'^2-\chi^2\kappa'^2)(\tau^2\theta'^2-\theta^2\tau'^2)}{(\varpi^2\beta'^2-\beta^2\varpi'^2)(\kappa^2\gamma'^2-\gamma^2\kappa'^2)(\tau^2\delta'^2-\delta^2\tau'^2)} \quad \ldots\ldots\ldots\ldots(43),$$

$$k'^2 = \frac{(\beta'^2-\beta^2)(\varpi^2\phi'^2-\phi^2\varpi'^2)(\kappa^2\chi'^2-\chi^2\kappa'^2)(\tau^2\theta'^2-\theta^2\tau'^2)}{(\phi'^2-\phi^2)(\varpi^2\beta'^2-\beta^2\varpi'^2)(\kappa^2\gamma'^2-\gamma^2\kappa'^2)(\tau^2\delta'^2-\delta^2\tau'^2)} \quad \ldots\ldots(44),$$

$$\frac{k^2\,\mathrm{cn}^2(q-t)}{k'^2} = \frac{(\varpi'^2-\varpi^2)(\phi^2\beta'^2-\beta^2\phi'^2)}{(\beta'^2-\beta^2)(\varpi^2\phi'^2-\phi^2\varpi'^2)} \quad \ldots\ldots\ldots\ldots(45).$$

We have, too,

$$\frac{\mathrm{cn}^2(q-t)\,\mathrm{dn}^2(q-t)}{\mathrm{sn}^2(q-t)} = \frac{\{\kappa^2k'^2+(1-\gamma^2)\,k^2m^2\}\{\tau^2k'^2+(1-\delta^2)\,k^2m^2\}}{\varpi^2k'^2+(1-\beta^2)\,k^2m^2}$$

$$= \frac{k'^2\{\kappa^2(\gamma'^2-\gamma^2)+(1-\gamma^2)(\kappa'^2-\kappa^2)\}\{\tau^2(\delta'^2-\delta^2)+(1-\delta^2)(\tau'^2-\tau^2)\}(\beta'^2-\beta^2)}{(\gamma'^2-\gamma^2)(\delta'^2-\delta^2)\{\varpi^2(\beta'^2-\beta)+(1-\beta^2)(\varpi'^2-\varpi^2)\}}$$

$$= \frac{k'^2(\beta'^2-\beta^2)(\chi^2\kappa'^2-\kappa^2\chi'^2)(\theta^2\tau'^2-\tau^2\theta'^2)}{(\gamma'^2-\gamma^2)(\delta'^2-\delta^2)(\phi^2\varpi'^2-\varpi^2\phi'^2)} \quad \ldots\ldots\ldots\ldots(46),$$

$$\mathrm{sn}^2(q-t) = \frac{(\phi'^2-\phi^2)(\gamma'^2-\gamma^2)(\delta'^2-\delta^2)(\beta^2\varpi'^2-\varpi^2\beta'^2)}{(\beta'^2-\beta^2)^2(\chi^2\gamma'^2-\gamma^2\chi'^2)(\theta^2\delta'^2-\delta^2\theta'^2)}$$

$$= \frac{(\chi'^2-\chi^2)(\theta'^2-\theta^2)(\beta^2\varpi'^2-\varpi^2\beta'^2)}{(\phi'^2-\phi^2)(\chi^2\gamma'^2-\gamma^2\chi'^2)(\theta^2\delta'^2-\delta^2\theta'^2)} \quad \ldots\ldots\ldots(47),$$

$$k^2\,\mathrm{sn}^2(q-t) = \frac{(\kappa'^2-\kappa^2)(\tau'^2-\tau^2)(\beta^2\phi'^2-\phi^2\beta'^2)}{(\varpi'^2-\varpi^2)(\kappa^2\gamma'^2-\gamma^2\kappa'^2)(\tau^2\delta'^2-\delta^2\tau'^2)} \quad \ldots\ldots\ldots(48).$$

14. Differentiating with respect to r, we have

$$\left.\begin{aligned}\frac{1}{\beta'}\frac{d\beta'}{dr} &= \frac{\operatorname{sn}(p+r)\operatorname{cn}(p+r)\operatorname{dn}(p+r)}{\operatorname{sn}^2(q-t)-\operatorname{sn}^2(p+r)}\\ \frac{1}{\varpi'}\frac{d\varpi'}{dr} &= \frac{\operatorname{sn}(p+r)\operatorname{dn}(p+r)\operatorname{cn}^2(q-t)}{\operatorname{cn}(p+r)\{\operatorname{sn}^2(q-t)-\operatorname{sn}^2(p+r)\}}\\ \frac{1}{\phi'}\frac{d\phi'}{dr} &= \frac{\operatorname{sn}(p+r)\operatorname{cn}(p+r)\operatorname{dn}^2(q-t)}{\operatorname{dn}(p+r)\{\operatorname{sn}^2(q-t)-\operatorname{sn}^2(p+r)\}}\end{aligned}\right\} \quad \ldots\ldots\ldots(49).$$

15. *Examination of case when* $\beta^2=\kappa^2=\theta^2=1$.

We have, from equations (25), (26),

$$\frac{k^2m^2}{n^2} = \frac{\operatorname{dn}(q-t)\operatorname{cn}(t-p)\operatorname{cn}(p-q)}{\operatorname{cn}(q-t)\operatorname{dn}(t-p)\operatorname{dn}(p-q)},$$

or

$$k^2 = \frac{\operatorname{dn}^2(q-t)}{\operatorname{cn}^2(q-t)},$$

showing that

$$\operatorname{sn}(q-t)=\infty.$$

In like manner, we have

$$0 = \frac{\operatorname{cn}(t-p)\operatorname{dn}(p-q)\operatorname{dn}(q-t)}{\operatorname{dn}(t-p)\operatorname{cn}(p-q)\operatorname{cn}(q-t)},$$

$$0 = \frac{\operatorname{dn}(p-q)\operatorname{cn}(q-t)\operatorname{cn}(t-p)}{\operatorname{cn}(p-q)\operatorname{dn}(q-t)\operatorname{dn}(t-p)},$$

showing that

$$\operatorname{cn}(t-p)=0,$$

$$\operatorname{dn}(p-q)=0.$$

No further relation is required. We must exclude the values of p which satisfy $\operatorname{sn}p=\infty$. So we may take

$$q = p+(2n+1)K+(2m+1)iK',$$

$$t = p+(2n'+1)K+2m'iK',$$

p having any value not included in the formula

$$2nK+2n'iK'+iK'.$$

In like manner, if we were to take

$$q = p+(2n+1)K+2miK',$$

$$t = p+(2n'+1)K+(2m'+1)iK',$$

we should find

$$\beta^2=\tau^2=\chi^2=1.$$

16. A special interest attaches to the case when

$$kk'=1.$$

We will take

$$k=-i\rho,\quad k'=i\rho^2,$$

where

$$\rho=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},$$

also $$\sqrt{\rho-1} = \frac{3^{\frac{1}{4}}\rho\,(1-i)}{2^{\frac{1}{2}}},$$

$$\sqrt{\rho^2-1} = \frac{3^{\frac{1}{4}}\rho^2\,(1+i)}{2^{\frac{1}{2}}};$$

having the real part positive. Now let

$$y^2 = \frac{\mathrm{sn}^2\sqrt{\rho-1}\,u}{\rho-1+\mathrm{sn}^2\sqrt{\rho-1}\,u} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50).$$

Then
$$\left.\begin{aligned} \mathrm{sn}^2\sqrt{\rho-1}\,u &= (\rho-1)\,y^2 \quad \div \\ \mathrm{cn}^2\sqrt{\rho-1}\,u &= 1-\rho y^2 \quad \div \\ \mathrm{dn}^2\sqrt{\rho-1}\,u &= 1-\rho^2y^2 \quad \div \\ \text{denominator} &= 1-y^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(51).$$

We suppose that, initially, $y = u$.

We easily find

$$-\sqrt{\rho-1}\,\mathrm{cn}\,.\,\mathrm{sn}\,.\,\mathrm{dn} = \frac{(1-\rho)\,y}{(1-y^2)^2}\,\frac{dy}{du},$$

$$\frac{du}{dy} = \frac{1}{\sqrt{1-y^6}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(52).$$

As y increases from zero, passing through real values, u increases from zero, passing through real values also, until $y=1$. The integral $\int_0^1 \frac{dy}{\sqrt{1-y^6}}$, is evaluated in Legendre's *Exercices.*

Also, y being always real, cn^2 and dn^2 in equations (51) are always imaginary, except when $y = 0$. So that, since $\mathrm{cn}\,0 = 1$, $\mathrm{dn}\,0 = 1$, in extracting the square root, such sign must be given to cn, dn as will make the real part of the square root positive. We may, in constructing the Table given on pages 214 and 215, first define $\frac{1}{3}\rho^2K$ by the equations

$$\left.\begin{aligned} \mathrm{sn}^2\tfrac{1}{3}\rho^2K &= 2^{\frac{2}{3}}-1 \quad \div \\ \mathrm{cn}^2\tfrac{1}{3}\rho^2K &= \rho^2-2^{\frac{2}{3}} \quad \div \\ \mathrm{dn}^2\tfrac{1}{3}\rho^2K &= 2^{\frac{2}{3}}\rho^2-1 \quad \div \\ \text{denominator} &= \rho^2-1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(53).$$

Then we must have

$$\mathrm{cn}\,\tfrac{1}{3}\rho^2K = \frac{3^{\frac{1}{4}}\,(1+i)}{2^{\frac{1}{2}}\,(2^{\frac{1}{3}}+\rho)}, \quad \mathrm{dn}\,\tfrac{1}{3}\rho^2K = \frac{3^{\frac{1}{4}}\,(1-i)}{2^{\frac{1}{2}}\,(2^{\frac{1}{3}}+\rho^2)} \quad \ldots\ldots\ldots(54).$$

For these have their real parts positive. It is to be understood that $\frac{1}{3}\rho^2K$ is the value obtained by $\sqrt{\rho-1}\,u$, by changing continuously as y changes continuously in increasing from zero to $\frac{2^{\frac{1}{3}}-1}{2^{\frac{2}{3}}}$. To fix the

sign of $\operatorname{sn} \frac{1}{3}\rho^2 K$, we take

$$\operatorname{sn} \tfrac{1}{3}\rho^2 K = -\frac{3^{\frac{1}{2}}(1-i)\rho}{2^{\frac{1}{3}}(2^{\frac{1}{3}}+1)} \quad \text{.....(55).}$$

We can now construct the Table by means of the addition equations, and the equations (13), (14), which now become

$$\left.\begin{aligned} \operatorname{sn}\rho u &= \frac{\rho \operatorname{sn} u}{\operatorname{dn} u}, \quad \operatorname{cn}\rho u = \frac{1}{\operatorname{dn} u}, \quad \operatorname{dn}\rho u = \frac{\operatorname{cn} u}{\operatorname{dn} u} \\ \operatorname{sn}\rho^2 u &= \frac{\rho^2 \operatorname{sn} u}{\operatorname{cn} u}, \quad \operatorname{cn}\rho^2 u = \frac{\operatorname{dn} u}{\operatorname{n} u}, \quad \operatorname{dn}\rho^2 u = \frac{1}{\operatorname{cn} u} \end{aligned}\right\} \text{.....(56).}$$

The following identities will be useful

$$\left.\begin{aligned} (2^{\frac{2}{3}}-1)(2^{\frac{1}{3}}+1)^2 &= 3 \\ (2^{\frac{2}{3}}-\rho^2)(2^{\frac{1}{3}}\rho+\rho^2)^2 &= 3 \\ (2^{\frac{2}{3}}-\rho)(2^{\frac{1}{3}}\rho^2+\rho)^2 &= 3 \\ \rho-1 &= -3^{\frac{1}{2}}i\rho^2 \\ \rho^2-1 &= \quad 3^{\frac{1}{2}}i\rho \end{aligned}\right\} \text{.....(57).}$$

It will be obvious that K has its usual meaning, and that

$$iK' = \rho^2 K \quad \text{.....(58).}$$

The Table could also be constructed by solving equations given by the triplication formulæ.

17. *Expressions for* $\operatorname{sn}(\rho-\rho^2)u$, *&c., when* $k = -i\rho$.

By means of equations (56), and the ordinary addition equations, we can easily find

$$\left.\begin{aligned} \operatorname{sn}(\rho-\rho^2)u &= \operatorname{sn} u\,(\operatorname{sn}^2 u+\rho-1) \;\div \\ \operatorname{cn}(\rho-\rho^2)u &= \operatorname{cn} u\,(\operatorname{sn}^2 u-\rho^2) \;\div \\ \operatorname{dn}(\rho-\rho^2)u &= -\operatorname{dn} u\,(\operatorname{sn}^2 u+\rho^2) \;\div \\ \text{denominator} &= (\rho^2-\rho)\operatorname{sn}^2 u-\rho^2 \end{aligned}\right\} \text{.....(59).}$$

$$\left.\begin{aligned} \operatorname{sn}(1-\rho)u &= \rho^2 \operatorname{sn} u\,(\operatorname{sn}^2 u+\rho-1) \;\div \\ \operatorname{cn}(1-\rho)u &= -\operatorname{dn} u\,(\operatorname{sn}^2 u+\rho^2) \;\div \\ \operatorname{dn}(1-\rho)u &= (\rho^2-\rho)\operatorname{sn}^2 u-\rho^2 \;\div \\ \text{denominator} &= \operatorname{cn} u\,(\operatorname{sn}^2 u-\rho^2) \end{aligned}\right\} \text{.....(60).}$$

$$\left.\begin{aligned} \operatorname{sn}(\rho^2-1)u &= -\rho \operatorname{sn} u\,(\operatorname{sn}^2 u+\rho-1) \;\div \\ \operatorname{cn}(\rho^2-1)u &= (\rho-\rho^2)\operatorname{sn}^2 u+\rho^2 \;\div \\ \operatorname{dn}(\rho^2-1)u &= -\operatorname{cn} u\,(\operatorname{sn}^2 u-\rho^2) \;\div \\ \text{denominator} &= \operatorname{dn} u\,(\operatorname{sn}^2 u+\rho^2) \end{aligned}\right\} \text{.....(61).}$$

18. *Elliptic Function Formulæ, when* $k = -i\rho$.

Besides the formulæ of Art. 17, there are numerous other formulæ of great interest. Thus equations (56) enable us to express rationally, in sn's only, the ordinary addition equations. As instances of formulæ

$k = -i\rho$.

A TABLE FOR sn, cn, dn, sn^2, cn^2, dn^2.

$u =$	$\mathrm{sn}\, u =$	$\mathrm{cn}\, u =$	$\mathrm{dn}\, u =$	$\mathrm{sn}^2 u =$	$\mathrm{cn}^2 u =$	$\mathrm{dn}^2 u =$
$\frac{1}{3}\rho^2 K$	$-\frac{3^{\frac{1}{4}}(1-i)\rho}{2^{\frac{1}{2}}(2^{\frac{1}{3}}+1)}$	$\frac{3^{\frac{1}{4}}(1+i)}{2^{\frac{1}{2}}(2^{\frac{1}{3}}+\rho)}$	$\frac{3^{\frac{1}{4}}(1-i)}{2^{\frac{1}{2}}(2^{\frac{1}{3}}+\rho^2)}$	$\frac{2^{\frac{2}{3}}-1}{\rho^2-1}$	$\frac{\rho^2-2^{\frac{2}{3}}}{\rho^2-1}$	$\frac{2^{\frac{2}{3}}\rho^2-1}{\rho^2-1}$
$\frac{1}{3}K$	$-\frac{2^{\frac{1}{3}}\rho^2+\rho}{2^{\frac{2}{3}}+1}$	$\frac{2^{\frac{1}{4}}(2^{\frac{1}{3}}+\rho^2)}{3^{\frac{1}{4}}(1-i)}$	$\frac{i\rho(2^{\frac{2}{3}}\rho-1)}{2^{\frac{2}{3}}+1}$	$\frac{2^{\frac{2}{3}}-1}{2^{\frac{2}{3}}-\rho}$	$\frac{1-\rho}{2^{\frac{2}{3}}-\rho}$	$\frac{1-2^{\frac{2}{3}}\rho}{2^{\frac{2}{3}}-\rho}$
$\frac{1}{3}\rho K$	$\frac{i(2^{\frac{1}{3}}+\rho)}{2^{\frac{2}{3}}+1}$	$-\frac{i\rho(2^{\frac{2}{3}}-\rho)}{2^{\frac{2}{3}}+1}$	$\frac{2^{\frac{1}{2}}(2^{\frac{1}{3}}+\rho)}{3^{\frac{1}{4}}(1+i)}$	$-\frac{(2^{\frac{2}{3}}-1)\rho}{2^{\frac{2}{3}}-\rho^2}$	$\frac{1-2^{\frac{2}{3}}\rho^2}{2^{\frac{2}{3}}-\rho^2}$	$\frac{1-\rho^2}{2^{\frac{2}{3}}-\rho^2}$
$\frac{1}{3}(1-\rho)K$	$-\frac{\rho(1+i)}{2^{\frac{1}{2}}.3^{\frac{1}{4}}}$	$-\frac{(1-i)\rho^2}{2^{\frac{1}{2}}.3^{\frac{1}{4}}}$	$-\frac{(1+i)\rho}{2^{\frac{1}{2}}.3^{\frac{1}{4}}}$	$\frac{1}{1-\rho^2}$	$\frac{\rho^2}{\rho^2-1}$	$\frac{1}{1-\rho^2}$
$\frac{1}{3}(\rho-\rho^2)K$	ρ	$-\frac{3^{\frac{1}{4}}(1-i)\rho^2}{2^{\frac{1}{2}}}$	$-i\rho$	ρ^2	$1-\rho^2$	$-\rho^2$
$\frac{1}{3}(\rho^2-1)K$	$i\rho$	$i\rho^2$	$-\frac{3^{\frac{1}{4}}(1+i)\rho}{2^{\frac{1}{2}}}$	$-\rho^2$	$-\rho$	$1-\rho$
$\frac{2}{3}\rho^2 K$	$-\frac{2^{\frac{1}{2}}\rho(2^{\frac{1}{3}}+1)}{3^{\frac{1}{4}}(1+i)}$	$-\frac{(2^{\frac{1}{3}}+1)\rho}{2^{\frac{1}{3}}+\rho^2}$	$-\frac{(2^{\frac{1}{3}}+1)\rho^2}{2^{\frac{1}{3}}+\rho}$	$\frac{\rho-1}{2^{\frac{2}{3}}-1}$	$\frac{2^{\frac{2}{3}}-\rho}{2^{\frac{2}{3}}-1}$	$\frac{2^{\frac{2}{3}}-\rho^2}{2^{\frac{2}{3}}-1}$
$\frac{2}{3}K$	$\frac{2^{\frac{1}{2}}(2^{\frac{1}{3}}+\rho)}{3^{\frac{1}{4}}(1+i)}$	$-\frac{(2^{\frac{1}{3}}+\rho)\rho}{2^{\frac{1}{3}}+1}$	$\frac{(2^{\frac{1}{3}}+\rho)\rho^2}{2^{\frac{1}{3}}+\rho^2}$	$\frac{1-\rho^2}{2^{\frac{2}{3}}-\rho^2}$	$\frac{2^{\frac{2}{3}}-1}{2^{\frac{2}{3}}-\rho^2}$	$\frac{2^{\frac{2}{3}}-\rho}{2^{\frac{2}{3}}-\rho^2}$
$\frac{2}{3}\rho K$	$\frac{2^{\frac{1}{2}}\rho^2(2^{\frac{1}{3}}+\rho^2)}{3^{\frac{1}{4}}(1+i)}$	$\frac{(2^{\frac{1}{3}}+\rho^2)\rho}{2^{\frac{1}{3}}+\rho}$	$-\frac{(2^{\frac{1}{3}}+\rho^2)\rho^2}{2^{\frac{1}{3}}+1}$	$\frac{\rho^2-\rho}{2^{\frac{2}{3}}-\rho}$	$\frac{2^{\frac{2}{3}}-\rho^2}{2^{\frac{2}{3}}-\rho}$	$\frac{2^{\frac{2}{3}}-1}{2^{\frac{2}{3}}-\rho}$

K	1	0	$i\rho^2$	1	0	$-\rho$
ρK	$-i\rho^2$	$-i\rho$	0	$-\rho$	$-\rho^2$	0
$\rho^2 K$	∞	$i\infty$	$\rho\infty$	∞^2	$-\infty^2$	$\rho^2\infty^2$
$\frac{2}{3}(1-\rho)K$	$-\frac{3^{\frac{1}{4}}(1+i)\rho}{2^{\frac{1}{2}}}$	$-\rho^2$	$-\rho$	$1-\rho$	ρ	ρ^2
$\frac{2}{3}(\rho-\rho^2)K$	$\frac{3^{\frac{1}{4}}(1+i)\rho}{2^{\frac{1}{2}}}$	$-\rho^2$	ρ	$1-\rho$	ρ	ρ^2
$\frac{2}{3}(\rho^2-1)K$	$\frac{3^{\frac{1}{4}}(1+i)\rho}{2^{\frac{1}{2}}}$	ρ^2	$-\rho$	$1-\rho$	ρ	ρ^2
$(\rho-\rho^2)K$	-1	0	$-i\rho^2$	1	0	$-\rho$
$(\rho^2-1)K$	$-i\rho^2$	$i\rho$	0	$-\rho$	$-\rho^2$	0
$(1-\rho)K$	∞	$i\infty$	$-\rho\infty$	∞^2	$-\infty^2$	$\rho^2\infty^2$
$\frac{4}{3}\rho^2 K$	$\frac{2^{\frac{1}{2}}(2^{\frac{1}{3}}+1)\rho}{3^{\frac{1}{4}}(1+i)}$	$\frac{(2^{\frac{1}{3}}+1)\rho}{2^{\frac{1}{3}}+\rho^2}$	$\frac{(2^{\frac{1}{3}}+1)\rho^2}{2^{\frac{1}{3}}+\rho}$	$\frac{\rho-1}{2^{\frac{2}{3}}-1}$	$\frac{2^{\frac{2}{3}}-\rho}{2^{\frac{2}{3}}-1}$	$\frac{2^{\frac{2}{3}}-\rho^2}{2^{\frac{2}{3}}-1}$
$\frac{4}{3}\rho K$	$\frac{2^{\frac{1}{2}}\rho^2(2^{\frac{1}{3}}+\rho^2)}{3^{\frac{1}{4}}(1+i)}$	$\frac{(2^{\frac{1}{3}}+\rho^2)\rho}{2^{\frac{1}{3}}+\rho}$	$\frac{(2^{\frac{1}{3}}+\rho^2)\rho^2}{2^{\frac{1}{3}}+1}$	$\frac{\rho^2-\rho}{2^{\frac{2}{3}}-\rho}$	$\frac{2^{\frac{2}{3}}-\rho^2}{2^{\frac{2}{3}}-\rho}$	$\frac{2^{\frac{2}{3}}-1}{2^{\frac{2}{3}}-\rho}$
$\frac{4}{3}K$	$\frac{2^{\frac{1}{2}}(2^{\frac{1}{3}}+\rho)}{3^{\frac{1}{4}}(1+i)}$	$\frac{(2^{\frac{1}{3}}+\rho)\rho}{2^{\frac{1}{3}}+1}$	$\frac{(2^{\frac{1}{3}}+\rho)\rho^2}{2^{\frac{1}{3}}+\rho^2}$	$\frac{1-\rho^2}{2^{\frac{2}{3}}-\rho^2}$	$\frac{2^{\frac{2}{3}}-1}{2^{\frac{2}{3}}-\rho^2}$	$\frac{2^{\frac{2}{3}}-\rho}{2^{\frac{2}{3}}-\rho^2}$

readily verified, we may notice

$$\left.\begin{aligned} \operatorname{cn} u \operatorname{cn} \rho u \operatorname{cn} \rho^2 u &= 1 \\ \operatorname{dn} u \operatorname{dn} \rho u \operatorname{dn} \rho^2 u &= 1 \end{aligned}\right\} \quad \ldots\ldots(62).$$

$$\rho^2 \operatorname{sn}^2 u - \operatorname{sn}^2 \rho u = \rho^2 \operatorname{sn}^2 u \operatorname{sn}^2 \rho u \quad \ldots\ldots(63).$$

$$\left.\begin{aligned} \operatorname{sn}(u+v) &= \frac{\operatorname{sn} \rho u \operatorname{sn} \rho^2 u \operatorname{sn} \rho v \operatorname{sn} \rho^2 v \,(\operatorname{sn}^2 u - \operatorname{sn}^2 v)}{\operatorname{sn} u \operatorname{sn} v \,(\operatorname{sn} v \operatorname{sn} \rho u \operatorname{sn} \rho^2 u - \operatorname{sn} u \operatorname{sn} \rho v \operatorname{sn} \rho^2 v)} \\ &= \frac{\operatorname{sn} u \operatorname{sn} v \,(\operatorname{sn} v \operatorname{sn} \rho u \operatorname{sn} \rho^2 u + \operatorname{sn} u \operatorname{sn} \rho v \operatorname{sn} \rho^2 v)}{\operatorname{sn} \rho u \operatorname{sn} \rho^2 u \operatorname{sn} \rho v \operatorname{sn} \rho^2 v \,(1 + \rho^2 \operatorname{sn}^2 u \operatorname{sn}^2 v)} \\ \operatorname{cn}(u+v) &= \frac{\operatorname{sn} u \operatorname{sn} \rho u \operatorname{sn} \rho^2 v - \operatorname{sn} v \operatorname{sn} \rho v \operatorname{sn} \rho^2 u}{\operatorname{sn} v \operatorname{sn} \rho u \operatorname{sn} \rho^2 u - \operatorname{sn} u \operatorname{sn} \rho v \operatorname{sn} \rho^2 v} \\ \operatorname{dn}(u+v) &= \frac{\operatorname{sn} u \operatorname{sn} \rho^2 u \operatorname{sn} \rho v - \operatorname{sn} v \operatorname{sn} \rho^2 v \operatorname{sn} \rho u}{\operatorname{sn} v \operatorname{sn} \rho u \operatorname{sn} \rho^2 u - \operatorname{sn} u \operatorname{sn} \rho v \operatorname{sn} \rho^2 v} \end{aligned}\right\} \quad \ldots(64).$$

$$\frac{\operatorname{sn}^2 u - \operatorname{sn}^2 v}{\operatorname{sn}^2 u \operatorname{sn}^2 v} = \frac{\operatorname{sn}^2 \rho u - \operatorname{sn}^2 \rho v}{\rho \operatorname{sn}^2 \rho u \operatorname{sn}^2 \rho v} \quad \ldots\ldots(65).$$

And the five formulæ given by the late Professor H. J. S. Smith in *Proceedings of the London Mathematical Society*, Vol. x., p. 97, give the following :—

When $x_1 + x_2 + x_3 + x_4 = 0$,

$$\rho^2 = \frac{1}{\operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{sn} x_3 \operatorname{sn} x_4} + \frac{1}{\operatorname{sn} \rho x_1 \operatorname{sn} \rho x_2 \operatorname{sn} \rho x_3 \operatorname{sn} \rho x_4} + \frac{1}{\operatorname{sn} \rho^2 x_1 \operatorname{sn} \rho^2 x_2 \operatorname{sn} \rho^2 x_3 \operatorname{sn} \rho^2 x_4} \quad \ldots\ldots(66).$$

$$\frac{1}{\operatorname{sn} \rho^2 x_1 \operatorname{sn} \rho^2 x_2} - \frac{1}{\operatorname{sn} \rho^2 x_3 \operatorname{sn} \rho^2 x_4} + \frac{\rho^2}{\operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{sn} \rho x_3 \operatorname{sn} \rho x_4} - \frac{\rho^2}{\operatorname{sn} \rho x_1 \operatorname{sn} \rho x_2 \operatorname{sn} x_3 \operatorname{sn} x_4} = 0 \quad \ldots\ldots(67),$$

$$\frac{1}{\operatorname{sn} x_2 \operatorname{sn} \rho x_3 \operatorname{sn} \rho^2 x_4} + \frac{1}{\operatorname{sn} x_3 \operatorname{sn} \rho x_2 \operatorname{sn} \rho^2 x_1} + \frac{1}{\operatorname{sn} x_4 \operatorname{sn} \rho x_1 \operatorname{sn} \rho^2 x_2} + \frac{1}{\operatorname{sn} x_1 \operatorname{sn} \rho x_4 \operatorname{sn} \rho^2 x_3} = 0 \quad \ldots\ldots(68).$$

The six formulæ of this type will give various simple expressions, by determinants, for $\operatorname{sn} x_1$ in terms of sn's of x_2, x_3, x_4, ρx_2, &c.

19. When $k = -i\rho$, our original equations, (1), (2), (3), become

$$\left.\begin{aligned} \beta^2 &= \frac{\rho \operatorname{cn}(q-t) \operatorname{dn}(q-t)}{\operatorname{sn}(t-p) \operatorname{sn}(p-q)\{\operatorname{sn}^2 p - \operatorname{sn}^2 (q-t)\}} \\ \varpi^2 &= \frac{\rho \operatorname{cn} \rho^2 (q-t) \operatorname{dn} \rho^2 (q-t)}{\operatorname{sn} \rho^2 (t-p) \operatorname{sn} \rho^2 (p-q) \{\operatorname{sn}^2 \rho^2 p - \operatorname{sn}^2 \rho^2 (q-t)\}} \\ \phi^2 &= \frac{\rho \operatorname{cn} \rho (q-t) \operatorname{dn} \rho (q-t)}{\operatorname{sn} \rho (t-p) \operatorname{sn} \rho (p-q)\{\operatorname{sn}^2 \rho p - \operatorname{sn}^2 \rho (q-t)\}} \end{aligned}\right\} \quad \ldots\ldots(69);$$

the other six formulæ being obtained, as in Art. 1, by a cyclical permutation of p, q, t.

20. The formulæ of Art. 5 give, when $k = -i\rho$, formulæ worthy of separate notice. Thus we have

$$\left.\begin{aligned} \frac{\beta\kappa\theta}{\gamma\tau\phi} &= \frac{\operatorname{cn}\rho p \operatorname{cn} q \operatorname{cn}\rho^2 t}{\operatorname{cn}\rho\,(q-t)\operatorname{cn}(t-p)\operatorname{cn}\rho^2(p-q)} \\ &= \frac{\operatorname{dn}(q-t)\operatorname{dn}\rho^2(t-p)\operatorname{dn}\rho(p-q)}{\operatorname{dn} p \operatorname{dn}\rho^2 q \operatorname{dn}\rho t} \\ \frac{\beta\tau\chi}{\gamma\varpi\theta} &= \frac{\operatorname{dn}\rho^2 p \operatorname{dn} q \operatorname{dn}\rho t}{\operatorname{dn}\rho^2(q-t)\operatorname{dn}(t-p)\operatorname{dn}\rho(p-q)} \\ &= \frac{\operatorname{cn}(q-t)\operatorname{cn}\rho(t-p)\operatorname{cn}\rho^2(p-q)}{\operatorname{cn} p \operatorname{cn}\rho q \operatorname{cn}\rho^2 t} \end{aligned}\right\} \quad \ldots\ldots\ldots(70).$$

$$\frac{\beta\kappa\theta \operatorname{sn} p \operatorname{sn}\rho^2 q \operatorname{sn}\rho t}{\operatorname{sn}(q-t)\operatorname{sn}\rho^2(t-p)\operatorname{sn}\rho(p-q)} = \frac{\gamma\tau\phi \operatorname{sn}\rho p \operatorname{sn} q \operatorname{sn}\rho^2 t}{\operatorname{sn}\rho(q-t)\operatorname{sn}(t-p)\operatorname{sn}\rho^2(p-q)}$$

$$= \frac{\delta\varpi\chi \operatorname{sn}\rho^2 p \operatorname{sn}\rho q \operatorname{sn} t}{\operatorname{sn}\rho^2(q-t)\operatorname{sn}\rho(t-p)\operatorname{sn}(p-q)} = \frac{\beta\tau\chi \operatorname{sn} p \operatorname{sn}\rho q \operatorname{sn}\rho^2 t}{\operatorname{sn}(q-t)\operatorname{sn}\rho(t-p)\operatorname{sn}\rho^2(p-q)}$$

$$= \frac{\gamma\varpi\theta \operatorname{sn}\rho^2 p \operatorname{sn} q \operatorname{sn}\rho t}{\operatorname{sn}\rho^2(q-t)\operatorname{sn}(t-p)\operatorname{sn}\rho(p-q)} = \frac{\delta\kappa\phi \operatorname{sn}\rho p \operatorname{sn}\rho^2 q \operatorname{sn} t}{\operatorname{sn}\rho(q-t)\operatorname{sn}\rho^2(t-p)\operatorname{sn}(p-q)} \quad \ldots\ldots\ldots\ldots(71).$$

21. *Examination of case when*

$$q = \rho p, \quad t = \rho^2 p, \quad k = -i\rho.$$

In this case we shall find

$$\left.\begin{aligned} \beta^2(p, \rho p, \rho^2 p, -i\rho) &= \frac{(\rho - \operatorname{sn}^4 p)^2}{4\operatorname{sn}^4 p\,(1-\rho-\operatorname{sn}^2 p)^2} \\ &= \kappa^2 = \theta^2 \\ \gamma^2(p, \rho p, \rho^2 p, -i\rho) &= \frac{(\rho - \operatorname{sn}^4 \rho p)^2}{4\operatorname{sn}^4 \rho p\,(1-\rho-\operatorname{sn}^2 \rho p)^2} \\ &= \tau^2 = \phi^2 \\ \delta^2(p, \rho p, \rho^2 p, -i\rho) &= \frac{(\rho - \operatorname{sn}^4 \rho^2 p)^2}{4\operatorname{sn}^4 \rho^2 p\,(1-\rho-\operatorname{sn}^2 \rho^2 p)^2} \\ &= \varpi^2 = \chi^2 \end{aligned}\right\} \quad \ldots\ldots\ldots(72).$$

For the details of the investigation, we have, from equations (59), (60), (61),

$$\operatorname{sn}^2 p - \operatorname{sn}^2(\rho - \rho^2)p = \operatorname{sn}^2 p \left\{ 1 - \frac{\operatorname{sn}^4 p + 2(\rho-1)\operatorname{sn}^2 p - 3\rho}{\rho - 2(\rho-1)\operatorname{sn}^2 p - 3\operatorname{sn}^4 p} \right\}$$

$$= \frac{4\rho \operatorname{sn}^2 p \operatorname{cn}^2 p \operatorname{dn}^2 p}{\{(\rho^2-\rho)\operatorname{sn}^2 p - \rho^2\}^2};$$

$$\frac{\operatorname{cn}(\rho-\rho^2)p \operatorname{dn}(\rho-\rho^2)p}{\operatorname{sn}(\rho^2-1)p \operatorname{sn}(1-\rho)p} = \frac{\operatorname{cn}^2 p\,(\operatorname{sn}^4 p - \rho)^2 \operatorname{dn}^2 p}{\{(\rho^2-\rho)\operatorname{sn}^2 p - \rho^2\}^2 \operatorname{sn}^2 p \{\operatorname{sn}^2 p + \rho - 1\}^2},$$

whence the expression for β^2.

To express γ^2, δ^2 in terms of $\operatorname{sn} p$, we have

$$\rho - \operatorname{sn}^4 \rho p = \left(\rho^2 - \frac{\rho^2 \operatorname{sn}^2 p}{1+\rho^2 \operatorname{sn}^2 p}\right)\left(\rho^2 + \frac{\rho^2 \operatorname{sn}^2 p}{1+\rho^2 \operatorname{sn}^2 p}\right)$$
$$= \frac{\rho\{1+(\rho^2-1)\operatorname{sn}^2 p\}\{1-\rho \operatorname{sn}^2 p\}}{(1+\rho^2 \operatorname{sn}^2 p)^2},$$
$$\operatorname{sn}^2 \rho p = \frac{\rho^2 \operatorname{sn}^2 p}{1+\rho^2 \operatorname{sn}^2 p},$$
$$1-\rho-\operatorname{sn}^2 \rho p = \frac{1-\rho-\operatorname{sn}^2 p}{1+\rho^2 \operatorname{sn}^2 p};$$

so that we have

$$\gamma^2(p, \rho p, \rho^2 p, -i\rho) = \frac{\{1+(\rho^2-1)\operatorname{sn}^2 p\}^2 (\rho^2 - \operatorname{sn}^2 p)^2}{4 \operatorname{sn}^4 p\,(1-\rho-\operatorname{sn}^2 p)^2} \quad \ldots\ldots(73),$$

$$\rho - \operatorname{sn}^4 \rho^2 p = \left(\rho^2 - \frac{\rho \operatorname{sn}^2 p}{1-\operatorname{sn}^2 p}\right)\left(\rho^2 + \frac{\rho \operatorname{sn}^2 p}{1-\operatorname{sn}^2 p}\right)$$
$$= \frac{\rho^2(\rho^2+\operatorname{sn}^2 p)\{1+(\rho^2-1)\operatorname{sn}^2 p\}}{(1-\operatorname{sn}^2 p)^2},$$
$$\operatorname{sn}^2 \rho^2 p = \frac{\rho \operatorname{sn}^2 p}{1-\operatorname{sn}^2 p},$$
$$1-\rho-\operatorname{sn}^2 \rho^2 p = \frac{1-\rho-\operatorname{sn}^2 p}{1-\operatorname{sn}^2 p},$$

so that we have

$$\delta^2(p, \rho p, \rho^2 p, -i\rho) = \frac{(1+\rho \operatorname{sn}^2 p)^2 \{1+(\rho^2-1)\operatorname{sn}^2 p\}^2}{4 \operatorname{sn}^4 p\,(1-\rho-\operatorname{sn}^2 p)^2} \ldots\ldots(74).$$

We are as yet at liberty to give what sign we please to β. If we take

$$\beta(p, \rho p, \rho^2 p, -i\rho) = \frac{\rho - \operatorname{sn}^4 p}{2 \operatorname{sn}^2 p\,(\operatorname{sn}^2 p - 1 + \rho)} \quad \ldots\ldots(75),$$

we must take

$$\left.\begin{aligned} \beta = \kappa = \theta \\ \text{and} \quad \gamma = \tau = \phi = \frac{\rho - \operatorname{sn}^4 \rho p}{2 \operatorname{sn}^2 \rho p\,(\operatorname{sn}^2 \rho p - 1 + \rho)} \\ \delta = \varpi = \chi = \frac{\rho - \operatorname{sn}^4 \rho^2 p}{2 \operatorname{sn}^2 \rho^2 p\,(\operatorname{sn}^2 \rho^2 p - 1 + \rho)} \end{aligned}\right\} \quad \ldots\ldots(76),$$

and we have

$$\beta + \tau + \chi = 1,$$
$$\kappa\theta - \chi\tau = \beta^2 - \chi\tau = \beta(\beta + \tau + \chi),$$

showing that the system is a positive system.

We may notice the formulæ

$$\left.\begin{aligned} \beta \operatorname{cn} p &= \rho\delta \operatorname{cn}(\rho - \rho^2) p \\ \gamma \operatorname{cn} \rho p &= \rho\beta \operatorname{cn}(\rho^2 - 1) p \\ \delta \operatorname{cn} \rho^2 p &= \rho\gamma \operatorname{cn}(1-\rho) p \end{aligned}\right\} \quad \ldots\ldots(77),$$

and
$$\left.\begin{aligned}\beta \operatorname{dn} p &= \rho^2\gamma \operatorname{dn}(\rho-\rho^2)p\\ \gamma \operatorname{dn}\rho p &= \rho^2\delta \operatorname{dn}(\rho^3-1)p\\ \delta \operatorname{dn}\rho^3 p &= \rho^2\beta \operatorname{dn}(1-\rho)p\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(78).$$

Had we taken β with the other sign, the sign of all the rest would have had to be changed, and we should have had a negative system.

22. *Examination of case when*

$$q = \rho^2 p, \quad t = \rho p, \quad k = -i\rho.$$

As the substitution of q for t and t for q changes the sign of Q in (10), it changes the sign of $\beta\kappa\theta$, $\beta\chi\tau$, &c. So we must take

$$\beta(p, \rho^2 p, \rho p, -i\rho) = -\beta(p, \rho p, \rho^2 p, -i\rho) \quad \ldots\ldots\ldots\ldots(79).$$

We observe, *en passant*, that, if we put $q = t$ in (1), we have

$$\beta^2(p, t, t, k) = \frac{1}{k^2 \operatorname{sn}^2 p \operatorname{sn}^2(p-t)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(80),$$

showing that, as we approach by continuous changes $\beta(p, t, t)$ in different directions, we get different signs.

We shall get
$$\left.\begin{aligned}\beta = \chi = \tau &= \frac{\operatorname{sn}^4 p - \rho}{2\operatorname{sn}^2 p(\operatorname{sn}^2 p - 1 + \rho)}\\ \gamma = \varpi = \theta &= \frac{\operatorname{sn}^4 \rho^2 p - \rho}{2\operatorname{sn}^2 \rho^2 p(\operatorname{sn}^2 \rho^2 p - 1 + \rho)}\\ \delta = \kappa = \phi &= \frac{\operatorname{sn}^4 \rho p - \rho}{2\operatorname{sn}^2 \rho p(\operatorname{sn}^2 \rho p - 1 + \rho)}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(81),$$

and
$$\beta + \kappa + \theta = -1, \quad \kappa\theta - \chi\tau = \beta,$$

showing that the system is still a positive system.

This case may be considered as identical with the case considered in Art. 21, changing the sign of every function, and interchanging gutturals and dentals.

23. *Examination of case when*

$$p + \rho q + \rho^2 t = 0\,; \quad k = -i\rho.$$

We shall have, see equations (70),

$$\operatorname{cn}\rho^2(q-t)\operatorname{cn}(t-p)\operatorname{cn}\rho(p-q) = \operatorname{cn}\rho^2(q-t)\operatorname{cn}\rho(q-t)\operatorname{cn}(q-t) = 1,$$
$$\operatorname{dn}\rho(q-t)\operatorname{dn}\rho^2(t-p)\operatorname{dn}(p-q) = 1\,;$$

$$\frac{\gamma\varpi\vartheta}{\delta\kappa\phi} = \operatorname{dn}\rho p \operatorname{dn}\rho^2 q \operatorname{dn} t,$$

$$\frac{\gamma\varpi\theta + \rho^2\beta\tau\chi}{\delta\kappa\phi} = \operatorname{dn}\rho p \operatorname{dn}\rho^2 q \operatorname{dn} t + \rho^2 \operatorname{cn}\rho p \operatorname{cn}\rho^2 q \operatorname{cn} t$$
$$= -\rho,$$

$$\rho^2(\beta\tau\chi-\gamma\varpi\theta)+\rho(\delta\kappa\phi-\gamma\varpi\theta)=0,$$

$$\rho^2(\theta^2-\beta^2)+\rho(\theta^2-\kappa^2)=0,$$

$$\beta^2+\rho^2\kappa^2+\rho\theta^2=0.$$

24. *Examination of case when*

$$p+\rho^2q+\rho t=0,\ k=-i\rho.$$

This case is precisely similar.

$$\operatorname{cn}\rho^2(q-t)\operatorname{cn}\rho(t-p)\operatorname{cn}(p-q)=1,$$

$$\frac{\gamma\tau\phi}{\delta\varpi\chi}=\operatorname{cn}\rho^2p\operatorname{cn}\rho q\operatorname{cn}t,$$

$$\frac{\beta\kappa\theta}{\delta\varpi\chi}=\operatorname{dn}\rho^2p\operatorname{dn}\rho q\operatorname{dn}t,$$

and

$$\beta\kappa\theta+\rho^2\gamma\tau\phi+\rho\delta\varpi\chi=0,$$

$$\beta^2+\rho^2\tau^2+\rho\chi^2=0.$$

25. The system of equations (1), (2), (3) suggested itself to me in this way. I supposed β, γ, &c. to satisfy equations (5) and (7), and to be functions of η, ϵ, a, where

$$\beta^2+\rho^2\kappa^2+\rho\theta^2=\eta,\quad \beta^2+\rho^2\tau^2+\rho\chi^2=\epsilon \qquad (82).$$

Then, differentiating, regarding η, ϵ as constant, I found

$$\left.\begin{aligned}
\frac{d\beta}{da}&=\rho A\varpi\phi\,(\kappa\tau-\rho\gamma\delta) & \frac{d\gamma}{da}&=\rho A\kappa\chi\,(\tau\varpi-\rho\delta\beta) & \frac{d\delta}{da}&=\rho A\tau\theta\,(\varpi\kappa-\rho\beta\gamma)\\
\frac{d\varpi}{da}&=A\phi\beta\,(\chi\theta-\rho\kappa\tau) & \frac{d\kappa}{da}&=A\chi\gamma\,(\theta\phi-\rho\tau\varpi) & \frac{d\tau}{da}&=A\theta\delta\,(\phi\chi-\rho\varpi\kappa)\\
\frac{d\phi}{da}&=\rho^2A\beta\varpi\,(\gamma\delta-\rho\chi\theta) & \frac{d\chi}{da}&=\rho^2A\gamma\kappa\,(\delta\beta-\rho\theta\phi) & \frac{d\theta}{da}&=\rho^2A\delta\tau\,(\beta\gamma-\rho\phi\chi)
\end{aligned}\right\} \qquad (83).$$

To represent the solution of this system of differential equations, the

cases when (5) and (7) are satisfied alone being considered, I took

$$\left.\begin{aligned} \mathrm{sn}^2\psi &= \frac{\beta^2(\phi^2-\rho\varpi^2)}{\phi^2(\beta^2-\rho^2\varpi^2)} \\ \mathrm{dn}^2\psi &= \frac{(\rho\phi^2-\beta^2)(\kappa\tau-\rho\gamma\delta)^2}{\phi^2(\chi^2-\rho\kappa^2)(\theta^2-\rho\tau^2)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(84).$$

which give

$$\left.\begin{aligned} k^2 &= \frac{\rho^2(\beta^2-\rho^2\varpi^2)(\gamma^2-\rho^2\kappa^2)(\delta^2-\rho^2\tau^2)}{(\phi^2-\rho\varpi^2)(\chi^2-\rho\kappa^2)(\theta^2-\rho\tau^2)} \\ k'^2 &= \frac{\rho(\phi^2-\rho^2\beta^2)(\chi^2-\rho^2\gamma^2)(\theta^2-\rho^2\delta^2)}{(\phi^2-\rho\varpi^2)(\chi^2-\rho\kappa^2)(\theta^2-\rho\tau^2)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(85);$$

so that we may, consistently, take

$$\left.\begin{aligned} \mathrm{sn}^2\xi &= \frac{\gamma^2(\chi^2-\rho\kappa^2)}{\chi^2(\gamma^2-\rho^2\kappa^2)} \\ \mathrm{dn}^2\xi &= \frac{(\rho\chi^2-\gamma^2)(\tau\varpi-\rho\delta\beta)^2}{\chi^2(\theta^2-\rho\tau^2)(\phi^2-\rho\varpi^2)} \\ \mathrm{sn}^2\zeta &= \frac{\delta^2(\theta^2-\rho\tau^2)}{\theta^2(\delta^2-\rho^2\tau^2)} \\ \mathrm{dn}^2\zeta &= \frac{(\rho\theta^2-\delta^2)(\varpi\kappa-\rho\beta\gamma)^2}{\theta^2(\phi^2-\rho\varpi^2)(\chi^2-\rho\kappa^2)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(86).$$

These equations lead to

$$\left.\begin{aligned} \left(\frac{d\xi}{da}\right)^2 &= -\rho^2A^2(\phi^2-\rho\varpi^2)(\chi^2-\rho\kappa^2)(\theta^2-\rho\tau^2) \\ &= \left(\frac{d\zeta}{da}\right)^2 = \left(\frac{d\psi}{da}\right)^2 \end{aligned}\right\} \quad \ldots\ldots\ldots(87),$$

and several interesting formulæ, such as

$$\mathrm{sn}^2(\psi-\xi) = \frac{\tau^2-\rho^2\theta^2}{(\beta^2-\rho^2\varpi^2)(\gamma^2-\rho^2\kappa^2)} \quad \ldots\ldots\ldots\ldots(88),$$

$$\mathrm{cn}^2(\psi-\xi) = \frac{\rho^2(\beta^2-\rho\phi^2)(\gamma^2-\rho\chi^2)}{(\beta^2-\rho^2\varpi^2)(\gamma^2-\rho^2\kappa^2)} \quad \ldots\ldots\ldots\ldots(89),$$

$$\mathrm{dn}^2(\psi-\xi) = \frac{(\beta^2-\rho\phi^2)(\gamma^2-\rho\chi^2)}{(\varpi^2-\rho^2\phi^2)(\kappa^2-\rho^2\chi^2)} \quad \ldots\ldots\ldots\ldots(90),$$

$$\mathrm{cn}(\psi-\xi)\,\mathrm{cn}(\xi-\zeta)\,\mathrm{cn}(\zeta-\psi) = \frac{\rho k'^2}{k^2} \quad \ldots\ldots\ldots\ldots(91),$$

$$\mathrm{dn}(\psi-\xi)\,\mathrm{dn}(\xi-\zeta)\,\mathrm{dn}(\zeta-\psi) = -\rho^2k'^2 \quad \ldots\ldots\ldots\ldots(92),$$

$$\left.\begin{aligned} \mathrm{sn}(\psi+\xi-\zeta) &= \frac{\rho}{k}\cdot\frac{\beta(\gamma\delta-\rho^2\kappa\tau)-\rho^2\varpi\phi}{\phi(\chi\theta-\rho\kappa\tau)+\rho\beta\varpi} \\ &= \frac{\rho}{k}\cdot\frac{\gamma(\delta\beta-\rho^2\tau\varpi)+\rho^2\kappa\chi}{\chi(\theta\phi-\rho\tau\varpi)-\rho\gamma\kappa} \end{aligned}\right\} \quad \ldots\ldots\ldots(93),$$

$$\left.\begin{aligned} \operatorname{cn}(\psi+\xi-\zeta) &= \frac{i\rho^2 k'}{k}\cdot\frac{\kappa(\rho\delta\beta-\tau\varpi)+\rho\gamma\chi}{\chi(\theta\phi-\rho^2\delta\beta)+\rho^2\gamma\kappa} \\ &= \frac{i\rho^2 k'}{k}\cdot\frac{\varpi(\rho\gamma\delta-\kappa\tau)-\rho\beta\phi}{\phi(\chi\theta-\rho^2\gamma\delta)-\rho^2\beta\varpi} \end{aligned}\right\} \quad \ldots\ldots\ldots(94),$$

$$\operatorname{dn}^2(\psi+\xi-\zeta) = \frac{(\rho\phi^2-\beta^2)(\rho\chi^2-\gamma^2)(\rho\theta^2-\delta^2)}{[\chi(\rho^2\tau\varpi-\rho\theta\phi)+\rho^2\gamma\kappa]^2} \quad \ldots\ldots\ldots(95),$$

$$\operatorname{sn}\psi = \frac{\rho\beta}{\phi}\cdot\frac{1}{k}\frac{\operatorname{dn}(\xi-\zeta)}{\operatorname{cn}(\xi-\zeta)} \quad \ldots\ldots\ldots(96),$$

$$\operatorname{cn}\psi = \frac{i\rho^2\varpi}{\phi}\cdot\frac{k'}{k}\frac{1}{\operatorname{cn}(\xi-\zeta)} \quad \ldots\ldots\ldots(97),$$

$$\operatorname{dn}\psi = \frac{i\rho(\rho\chi\theta-\rho^2\kappa\tau)}{\phi}\frac{k'\operatorname{sn}(\xi-\zeta)}{\operatorname{cn}(\xi-\zeta)} \quad \ldots\ldots\ldots(98);$$

and, making use of (91), (92) to obtain formulæ from which ρ, ρ^2 should be absent, I got a system of equations resembling (21), that is to say,

$$\frac{\operatorname{sn}^2\psi}{\operatorname{cn}^2\psi} = \frac{\beta^2}{\varpi^2}\cdot\frac{\operatorname{dn}(\xi-\zeta)}{\operatorname{dn}(\psi-\xi)\operatorname{dn}(\zeta-\psi)},$$

$$\operatorname{cn}^2\psi = -\frac{\varpi^2}{\phi^2}\cdot\frac{\operatorname{cn}(\psi-\xi)\operatorname{cn}(\zeta-\psi)}{\operatorname{cn}(\xi-\zeta)},$$

in which β, γ, &c. satisfy (5), (7), without any such condition being necessary as is contained in equations (91), (92). This suggested the whole investigation.

On Mr. Wilkinson's Rectangular Transformation.

By Professor Cayley, F.R.S.

[*Read May 10th*, 1883.]

Considering the three cones,

$$(p+\lambda)X^2+(q+\lambda)Y^2+(r+\lambda)Z^2 = 0,$$
$$(p+\mu)X^2+(q+\mu)Y^2+(r+\mu)Z^2 = 0,$$
$$(p+\nu)X^2+(q+\nu)Y^2+(r+\nu)Z^2 = 0,$$

where

$$p+q+r+\lambda+\mu+\nu = 0,$$

it is easy to see that these contain a singly infinite system of rectangular axes, viz., we have in each cone one axis of a rectangular system, and for one of the cones the axis may be any line at pleasure of the cone. In fact, taking for the three axes (x, y, z), (x', y', z'),

(x'', y'', z'') respectively (that is, for the first axis $X : Y : Z = x : y : z$, and so for each of the other two axes), then (x, y, z) being an arbitrary line on the first cone, we can find (x', y', z') and (x'', y'', z'') such that

$$\begin{aligned}
(p+\lambda)\,x^2 + (q+\lambda)\,y^2 + (r+\lambda)\,z^2 &= 0,\\
(p+\mu)\,x'^2 + (q+\mu)\,y'^2 + (r+\mu)\,z'^2 &= 0,\\
(p+\nu)\,x''^2 + (q+\nu)\,y''^2 + (r+\nu)\,z''^2 &= 0,\\
x'x'' + y'y'' + z'z'' &= 0,\\
x''x + y''y + z''z &= 0,\\
x\,x' + y\,y' + z\,z' &= 0.
\end{aligned}$$

For, eliminating (x'', y'', z'') from the third, fourth, and fifth equations, we have, first,

$$x'' : y'' : z'' = yz' - y'z : zx' - z'x : xy' - x'y,$$

and consequently

$$(p+\nu)(yz'-y'z)^2 + (q+\nu)(zx'-z'x)^2 + (r+\nu)(xy'-x'y)^2 = 0,$$

and it is to be shown that this equation is implied in the remaining first, second, and third equations; for, this being so, (x, y, z), (x', y', z') satisfy only these equations; or (x, y, z) are any values whatever satisfying the first equation. The other two equations then determine (x', y', z'), and, these being known, (x'', y'', z'') are then determined as above.

In fact, attending to the sixth equation, the equation just obtained may be written in the form

$$\begin{aligned}
(p+\nu)\,[(y^2+z^2)(y'^2+z'^2) - x^2x'^2] + (q+\nu)\,[(z^2+x^2)(z'^2+x'^2) - y^2y'^2]\\
+ (r+\nu)\,[(x^2+y^2)(x'^2+y'^2) - z^2z'^2] = 0,
\end{aligned}$$

or, what is the same thing, in the form

$$\begin{aligned}
-(x^2+y^2+z^2)\,[(p+\mu)\,x'^2 + (q+\mu)\,y'^2 + (r+\mu)\,z'^2]\\
-(x'^2+y'^2+z'^2)\,[(p+\lambda)\,x^2 + (q+\lambda)\,y^2 + (r+\lambda)\,z^2] = 0\,;
\end{aligned}$$

for, comparing in the two forms, first the coefficients of $x^2x'^2$, these are

$$(q+\nu)+(r+\nu)-(p+\nu) \quad \text{and} \quad -2p+\lambda+\mu,$$

which are equal in virtue of $p+q+r+\lambda+\mu+\nu = 0$; and comparing next the coefficients of $y^2z'^2$, these are

$$p+\nu \quad \text{and} \quad -(r+\mu)-(q+\lambda),$$

which are equal in virtue of the same relation: and similarly the coefficients of the other terms $y^2y'^2$, &c. are equal in the two equations respectively.

Take now three arguments a_0, b_0, c_0, connected by the relation $a_0+b_0+c_0 = 0$, and write a, a, A for the sn, cn, and dn of a_0; and similarly b, b, B and c, c, C for those of b_0 and c_0 respectively: then

we may write

$$p+\lambda,\ q+\lambda,\ r+\lambda = \left(1,\ \frac{a}{bc},\ \frac{A}{BC}\right),$$

$$p+\mu,\ q+\mu,\ r+\mu = \theta\left(1,\ \frac{b}{ca},\ \frac{B}{CA}\right),$$

$$p+\nu,\ q+\nu,\ r+\nu = \phi\left(1,\ \frac{c}{ab},\ \frac{C}{AB}\right);$$

for, starting from the first set of values, we have the second set if only

$$\mu-\lambda = \theta-1 = \theta\frac{b}{ca}-\frac{a}{bc} = \theta\frac{B}{CA}-\frac{A}{BC}.$$

We thence obtain

$$\theta\left(1-\frac{b}{ca}\right) = \left(1-\frac{a}{bc}\right),\quad \theta\left(1-\frac{B}{CA}\right) = 1-\frac{A}{BC};$$

and, in order to the identity of the two values of θ, we must have

$$\left(1-\frac{b}{ca}\right)\left(1-\frac{A}{BC}\right)-\left(1-\frac{a}{bc}\right)\left(1-\frac{B}{CA}\right) = 0,$$

that is, $(abc-b^3)(ABC-A^3)-(abc-a^3)(ABC-B^3) = 0,$

or, reducing,

$$(a^3-b^3)\,ABC-(A^3-B^3)\,abc+A^3b^3-B^3a^3 = 0.$$

But $a^3-b^3 = -(a^3-b^3)$, $A^3-B^3 = -k^3(a^3-b^3)$, $A^3b^3-Ba^3 = k^3(a^3-b^3)$; hence the whole equation divides by a^3-b^3, and, omitting this factor, it becomes

$$-ABC+k^3abc+k^3 = 0,$$

which is a known relation between the elliptic functions of the arguments a_0, b_0, c_0 connected by the equation $a_0+b_0+c_0 = 0$. Similarly, for ϕ, we have

$$\nu-\lambda = \phi-1 = \phi\frac{c}{ab}-\frac{a}{bc} = \phi\frac{C}{AB}-\frac{A}{BC},$$

and, comparing the two values of ϕ, we have the same identical relation.

It thus appears that the three cones

$$X^3+\frac{a}{bc}\,Y^3+\frac{A}{BC}\,Z^3 = 0,$$

$$X^3+\frac{b}{ca}\,Y^3+\frac{B}{CA}\,Z^3 = 0,$$

$$X^3+\frac{c}{ab}\,Y^3+\frac{C}{AB}\,Z^3 = 0,$$

(the coefficients whereof depend on the elliptic functions sn, cn, and dn, of the arguments a_0, b_0, c_0 connected by the equation $a_0+b_0+c_0=0$) contain a singly infinite system of rectangular axes.

Considering an argument f_0, and denoting its sn, cn, dn by f, f, f' respectively, we have, for an arbitrary line on the first cone, the values

$$x,\ y,\ z = M\sqrt{k'^2 A\mathrm{a}},\quad M\sqrt{k^2 A\mathrm{bc}}\,.\,\mathrm{f},\quad M\sqrt{-\mathrm{a}BC}\,.\,F.$$

In fact, substituting in the equation of the cone, we obtain the identity $k'^2+k^2\mathrm{f}^2-F^2=0$; and if we determine M by the condition that $x^2+y^2+z^2$ shall be $=1$, then we have

$$1 = M^2\{k'^2 A\mathrm{a}+k^2 A\mathrm{bcf}^2-\mathrm{a}BCF^2\},$$

where the coefficient of M^2 is

$$= k'^2 A\mathrm{a}+k^2 A\mathrm{bc}\,(1-f^2)-\mathrm{a}BC\,(1-k^2f^2),$$

which is easily shown to be

$$= k^2k'^2\,.\,bc\,(a^2-f^2),$$

so that the values of x, y, z are

$$= \{\sqrt{k'^2 A\mathrm{a}},\quad \sqrt{k^2 A\mathrm{bc}}\,.\,\mathrm{f},\quad \sqrt{-\mathrm{a}BC}\,.\,F\} \div \sqrt{k^2k'^2\,.\,bc\,(a^2-f^2)},$$

and, similarly taking the arguments g_0, h_0, and denoting their elliptic functions by g, g, G, h, h, H, we have for a system of arbitrary lines in the three cones respectively, the values

$$\begin{array}{lcccc}
x,\ y,\ z = & \sqrt{k'^2 A\mathrm{a}} & \sqrt{k^2 A\mathrm{bc}}\,.\,f & \sqrt{-\mathrm{a}BC}\,.\,F & \div\sqrt{k^2k'^2\,.\,bc\,(a^2-f^2)} \\
x',\ y',\ z' = & \sqrt{k'^2 B\mathrm{b}} & \sqrt{k^2 B\mathrm{ca}}\,.\,g & \sqrt{-\mathrm{b}CA}\,.\,G & \div\sqrt{k^2k'^2\,.\,ca\,(b^2-g^2)} \\
x'',\ y'',\ z'' = & \sqrt{k'^2 C\mathrm{c}} & \sqrt{k^2 C\mathrm{ab}}\,.\,h & \sqrt{-\mathrm{c}AB}\,.\,H & \div\sqrt{k^2k'^2\,.\,ab\,(c^2-h^2)},
\end{array}$$

these values being such that $x^2+y^2+z^2$, $x'^2+y'^2+z'^2$, $x''^2+y''^2+z''^2$ are each $=1$. The radicals in the first line would be more correctly written, and may be understood as meaning $k'\sqrt{A}\sqrt{\mathrm{a}}$, $k\sqrt{A}\sqrt{\mathrm{b}}\sqrt{\mathrm{c}}$, $i\sqrt{\mathrm{a}}\sqrt{B}\sqrt{C}$, and similarly as regards the second and third lines respectively.

Taking now the arbitrary lines at right angles to each other, the condition for the second and third lines is

$$1+k^2\mathrm{agh}-k'^2AGH=0,$$

which is satisfied if $a_0=g_0-h_0$; similarly the condition for the third and first lines is satisfied if $b_0=h_0-f_0$; and we then have $a_0+b_0=g_0-f_0$; that is, $-c_0=g_0-f_0$ or $c_0=f_0-g_0$, which is the condition for the first and second lines; hence the arguments a_0, b_0, c_0, f_0, g_0, h_0 being such that

$$\begin{array}{rcl}
. & h_0-g_0+a_0 & =0, \\
-h_0 & . +f_0+b_0 & =0, \\
g_0-f_0 & . +c_0 & =0, \\
-a_0-b_0-c_0 & . & =0,
\end{array}$$

or, what is the same thing, $a_0, b_0, c_0, f_0, g_0, h_0$ being the differences of any four arguments $\alpha, \beta, \gamma, \delta$, the foregoing values of (x, y, z), (x', y', z'), (x'', y'', z'') will satisfy the equations

$$\begin{aligned} x^2 + y^2 + z^2 &= 1, \\ x'^2 + y'^2 + z'^2 &= 1, \\ x''^2 + y''^2 + z''^2 &= 1, \\ x'x'' + y'y'' + z'z'' &= 0, \\ x''x + y''y + z''z &= 0, \\ xx' + yy' + zz' &= 0, \end{aligned}$$

for the transformation of a set of rectangular axes. These are, in fact, Mr. Wilkinson's expressions, the $a_0, b_0, c_0, f_0, g_0, h_0$ being his $t-p$, $p-q$, $q-t$, t, p, q respectively.

Returning to the three cones, it is to be remarked that, taking in the first of them a line 1 at pleasure, then we have in the second of them *two* lines 2, 2′ each at right angles to the line 1, and such that the line 3 at right angles to the plane 12, and the line 3′ at right angles to the plane 12′, lie each of them in the third cone; or, what is the same thing we have in the two cones respectively the rectangular lines 1 and 2, and also the rectangular lines 1 and 2′, such that the planes 12 and 12′ each of them envelope one and the same cone, the reciprocal of the third cone; where by the reciprocal cone of a given cone is meant the cone generated by the lines through the vertex at right angles to the tangent planes of the given cone. Introducing the notion of the absolute cone $X^2 + Y^2 + Z^2 = 0$, a line and plane through the vertex at right angles to each other are, in fact, reciprocal polars in regard to this absolute cone; and two lines at right angles to each other are reciprocals (or harmonics) in regard to this absolute cone; that is, the reciprocal plane of either of them passes through the other. The two cones are cones intersecting each other in four lines lying on the absolute cone; and in virtue of this relation they have the property in question, viz., taking in the first cone a line 1 at pleasure, then the reciprocal plane hereof in regard to the absolute cone meets the second cone in a pair of lines 2 and 2′ such that the planes 12 and 12′ each of them envelope one and the same cone; the reciprocal of this cone is then the third cone of the system, and as such it passes through the four lines on the absolute cone.

In verification, observe that the coefficients $p+\lambda$, $q+\lambda$, &c. of the equations of the three cones satisfy the equations

$$\begin{Vmatrix} 1, & p+\lambda, & p+\mu, & p+\nu, & (p+\lambda)(p+\mu)(p+\nu) \\ 1, & q+\lambda, & q+\mu, & q+\nu, & (q+\lambda)(q+\mu)(q+\nu) \\ 1, & r+\lambda, & r+\mu, & r+\nu, & (r+\lambda)(r+\mu)(r+\nu) \end{Vmatrix} = 0.$$

This is obviously the case for each equation such as

$$| 1,\ p+\lambda,\ p+\mu | = 0;$$

and any equation containing the fifth column is at once reducible to

$$| 1,\ p,\ p^3+p^2(\lambda+\mu+\nu) | = 0,$$

that is, $$| 1,\ p,\ p^3, | + (\lambda+\mu+\nu) | 1,\ p,\ p^2 | = 0;$$

or, dividing by $| 1, p, p^2 |$, this is $p+q+r+\lambda+\mu+\nu=0$, the equation connecting the coefficients.

Hence, representing the three cones by

$$pX^2+qY^2+rZ^2=0,$$
$$p'X^2+q'Y^2+r'Z^2=0,$$
$$p''X^2+q''Y^2+r''Z^2=0,$$

and the absolute by $$X^2+Y^2+Z^2=0,$$

the coefficients p, q, &c. are connected by the equations

$$\left\| \begin{matrix} 1, & p, & p', & p'', & pp'p'' \\ 1, & q, & q', & q'', & qq'q'' \\ 1, & r, & r', & r'', & rr'r'' \end{matrix} \right\| = 0;$$

among these are of course included the equation $| 1, p, p' | = 0$, which expresses that the first and second cones intersect on the absolute; (p, q, r), (p', q', r') are any quantities satisfying this relation, and, regarding them as given, we have then two independent equations determining the ratios $p'' : q'' : r''$. The theorem is that the planes 12 and 12′ envelope one and the same quadric cone

$$\frac{X^2}{p''}+\frac{Y^2}{q''}+\frac{Z^2}{r''}=0.$$

The equations $| 1, p, p'' | = 0$ and $| 1, p, pp'p'' | = 0$ give

$$(q-r)\ p''\ +(r-p)\ q''\ +(p-q)\ r''\ =0,$$
$$(q-r)\ pp'p''+(r-p)\ qq'q''+(p-q)\ rr'r''=0,$$

and thence

$$(q-r)\ p'' : (r-p)\ q'' : (p-q)\ r''= qq'-rr' : rr'-pp' : pp'-qq';$$

or, observing that we have

$$q-r : r-p : p-q = qr'-q'r : rp'-r'p : pq'-p'q,$$

the equations may also be written

$$(qr'-q'r)\ p'' : (rp'-r'p)\ q'' : (pq'-p'q)\ r'' = qq'-rr' : rr'-pp' : pp'-qq'.$$

Starting with an arbitrary line (x, y, z) in the first cone, then the reciprocal plane thereof (in regard to the absolute cone) is the plane $Xx+Yy+Zz=0$, which meets the second cone in two lines, say (2)

and (2′), each of which is a line reciprocal to the line (1); and we have thus two planes (12) and (12′), each of which envelopes, as is to be shown, the same cone $q''r''X^2+r''p''Y^2+p''q''Z^2=0$.

Suppose, in general, that we have an arbitrary line (x, y, z) and an arbitrary plane $\alpha X+\beta Y+\gamma Z=0$, and that it is required to find the equation of the two planes through the line (x, y, z), and the intersections of the plane $\alpha X+\beta Y+\gamma Z=0$ with the cone $p'X^2+q'Y^2+r'Z^2=0$: the equation of the pair of planes is

$$(\alpha X+\beta Y+\gamma Z)^2(p'x^2+q'y^2+r'z^2)$$
$$+(\alpha x+\beta y+\gamma z)^2(p'X^2+q'Y^2+r'Z^2)$$
$$-2(\alpha X+\beta Y+\gamma Z)(\alpha x+\beta y+\gamma z)(p'Xx+q'Yy+r'Zz)=0.$$

In the present case, the plane $\alpha X+\beta Y+\gamma Z=0$ is the plane $xX+yY+zZ=0$, which is the reciprocal of the line (x, y, z) in regard to the absolute cone, and the equation of the pair of planes is

$$(xX+yY+zZ)^2(p'x^2+q'y^2+r'z^2)$$
$$+(x^2+y^2+z^2)^2(p'X^2+q'Y^2+r'Z^2)$$
$$-2(xX+yY+zZ)(x^2+y^2+z^2)(p'Xx+q'Yy+r'Zz)=0,$$

where the (x, y, z), as belonging to a line on the first cone, satisfy the condition $px^2+qy^2+rz^2=0$. The equation may be written

$$(a, b, c, f, g, h \mathbin{\text{\textbackslash\!\!\textbackslash}} yZ-zY, zX-xZ, xY-yX)^2=0,$$

where

$$a, b, c, f, g, h = q'z^2+r'y^2,\ r'x^2+p'z^2,\ p'y^2+q'x^2,\ -p'yz,\ -q'zx,\ -r'xy,$$

and, as before, $px^2+qy^2+rz^2=0$; viz., this is the equation of the pair of planes (12) and (12′).

The equation of the pair of tangent planes through the line (x, y, z) to the cone $q''r''X^2+r''p''Y^2+p''q''Z^2=0$ is

$$(q''r''x^2+r''p''y^2+p''q''z^2)(q''r''X^2+r''p''Y^2+p''q''Z^2)$$
$$-(q''r''xX+r''p''yY+p''q''zZ)^2=0;$$

viz., omitting a factor $p''q''r''$, this equation is

$$(p'', q'', r'', 0, 0, 0 \mathbin{\text{\textbackslash\!\!\textbackslash}} yZ-zY, zX-xZ, xY-yX)^2=0.$$

And it is to be shown that this is equivalent to the former equation; viz., writing $yZ-zY,\ zX-xZ,\ xY-yX=\lambda, \mu, \nu$, then that the two equations

$$(q'z^2+r'y^2,\ r'x^2+p'z^2,\ p'y^2+q'x^2,\ -p'yz,\ -q'zx,\ -r'xy \mathbin{\text{\textbackslash\!\!\textbackslash}} \lambda, \mu, \nu)^2=0,$$
$$p''\lambda^2+q''\mu^2+r''\nu^2=0$$

are equivalent to each other.

We have identically $\lambda x+\mu y+\nu z=0$, and thence also

$$(\lambda x+\mu y+\nu z)[(p'-q'-r')\lambda x+(-p'+q'-r')\mu y+(-p'-q'+r')\nu z]=0,$$

where, on the left-hand side, the terms in $\mu\nu$, $\nu\lambda$, and $\lambda\mu$ are

$= -2p'yz\,\mu\nu - 2q'zx\,\nu\lambda - 2r'xy\,\lambda\mu$. Hence the first equation may be written

$$[q'z^2+r'y^2+(p'-q'-r')\,x^2]\,\lambda^2+[r'x^2+p'z^2+(-p'+q'-r')\,y^2]\,\mu^2 + [p'y^2+q'x^2+(-p'-q'+r')\,z^2]\,\nu^2 = 0,$$

and it is to be shown that this is equivalent to

$$p''\lambda^2+q''\mu^2+r''\nu^2 = 0;$$

viz., that we have $p'' : q'' : r'' =$

$$\begin{aligned} & q'z^2+r'y^2-(\ \ p'-q'-r')\,x^2 \\ : & \ r'x^2+p'z^2-(-p'+q'-r')\,y^2 \\ : & \ p'y^2+q'x^2-(-p'-q'+r')\,z^2, \end{aligned}$$

where $px^2+qy^2+rz^2 = 0$. Writing the equation in the form

$$p'' : q'' : r'' = A : B : C,$$

we have

$$\begin{aligned} A &= q'z^2+r'y^2-p'x^2+q'x^2+r'x^2 \\ &= -p'x^2+q'(x^2+z^2)+r'(x^2+y^2) \\ &= -p'x^2+(q'+r')(x^2+y^2+z^2)-q'y^2-r'z^2, \end{aligned}$$

and by what precedes we have an identity of the form

$$x^2+y^2+z^2 = \alpha(p'x^2+q'y^2+r'z^2)+\beta(px^2+qy^2+rz^2),$$

where, determining α from the equations $1 = q'\alpha+q\beta$, $1 = r'\alpha+r\beta$, we find $\alpha = (q-r)\div(qr'-q'r)$; but $px^2+qy^2+rz^2 = 0$, and the relation thus is $x^2+y^2+z^2 = \alpha(p'x^2+q'y^2+r'z^2)$; hence

$$A = \{(q'+r')\,\alpha-1\}(p'x^2+q'y^2+r'z^2),$$

or, substituting for α its value, this is

$$\begin{aligned} A &= \left\{\frac{(q'+r')(q-r)-qr'+q'r}{qr'-q'r}\right\}(p'x^2+q'y^2+r'z^2), \\ &= \frac{qq'-rr'}{qr'-q'r}(p'x^2+q'y^2+r'z^2); \end{aligned}$$

and, forming the like values of B and C, the relations to be verified become

$$p'' : q'' : r'' = \frac{qq'-rr'}{qr'-q'r} : \frac{rr'-pp'}{rp'-r'p} : \frac{pp'-qq'}{pq'-p'q},$$

which are, in fact, the values of the ratios $p'' : q'' : r''$ obtained above; and the theorem is thus seen to be true. It may be remarked that, if the first and second cones, instead of intersecting in four lines on the absolute cone, had been arbitrary cones; then, taking in the first cone a line (1) and in the second cone a line (2), the reciprocal of (1) in regard to the absolute, the envelope of the plane (12) would have been (instead of a quadric cone) a cone of the class 8,

On the Motion of a Particle on the Surface of an Ellipsoid.

By W. R. Westropp Roberts.

[*Read May 10th*, 1883.]

1. The cases of motion of a particle constrained to move on the surface of an ellipsoid discussed in this paper are those in which the velocity, expressed as a function of the primary axes of the confocal hyperboloids determining the position of the particle on the surface, assumes a certain form.

The following theorem in the Calculus of Variations, which can be easily established, enables us to determine the motion and path of the particle in the cases considered.

If $\int \sqrt{\Phi(x)+\Psi(y)}\ \sqrt{dx^2+dy^2}$ be a minimum, then

$$\frac{dx}{\sqrt{\Phi(x)-C}} \pm \frac{dy}{\sqrt{\Psi(y)+C}} = 0$$

is a first integral of the differential equation afforded by the Calculus of Variations, C being an arbitrary constant; and hence, more generally,

if $\int \sqrt{\Phi(x)+\Psi(y)}\ \sqrt{\phi(x)\,dx^2+\psi(y)\,dy^2}$

be a minimum, then the first equation becomes

$$\frac{\sqrt{\phi(x)}\,.\,dx}{\sqrt{\Phi(x)-C}} \pm \frac{\sqrt{\psi(y)}\,.\,dy}{\sqrt{\Psi(y)+C}} = 0.$$

2. Let us now express ds the element of the arc of a curve on the surface of the ellipsoid in terms of μ and ν, the primary axes of the confocal surfaces through the point considered (see Salmon's *Geometry of Three Dimensions*, Art. 424).

Then $$ds^2 = d\sigma^2 + d\sigma'^2,$$

where $$d\sigma = \sqrt{\frac{(\mu^2-\nu^2)(a^2-\mu^2)}{(\mu^2-h^2)(k^2-\mu^2)}}\,d\mu \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$d\sigma' = \sqrt{\frac{(\mu^2-\nu^2)(a^2-\nu^2)}{(h^2-\nu^2)(k^2-\nu^2)}}\,d\nu.$$

The Principle of Least Action gives us $\int v\,ds$ a minimum, or

$$\int v\sqrt{\mu^2-\nu^2}\sqrt{\frac{(a^2-\mu^2)\,d\mu^2}{(\mu^2-h^2)(k^2-\mu^2)} + \frac{(a^2-\nu^2)\,d\nu^2}{(h^2-\nu^2)(k^2-\nu^2)}}$$

a minimum for the motion of the particle on the surface; hence we see that, if $v\sqrt{\mu^2-\nu^2} \equiv \sqrt{\Phi(\mu)+\Psi(\nu)}$,

the dynamical problem is reduced to the mathematical problem already discussed. If then

$$v^2 = \frac{\Phi(\mu)+\Psi(\nu)}{\mu^2-\nu^2},$$

$\Phi(\mu)$ and $\Psi(\nu)$ involving the coordinates μ' and ν' of the initial position, and β the initial velocity of the particle, the differential equation of the path will be

$$\sqrt{\frac{(a^2-\mu^2)\,.\,d\mu}{(\mu^2-h^2)(k^2-\mu^2)\{\Phi(\mu)-C\}}} \pm \frac{\sqrt{a^2-\nu^2}\,.\,d\nu}{\sqrt{(h^2-\nu)(k^2-\nu)}\,\{\Psi(\nu)+C\}} = 0 \ldots (2),$$

or multiplying by $\sqrt{\mu^2-\nu^2}$, and referring to (1),

$$\frac{d\sigma}{\sqrt{\Phi(\mu)-C}} \pm \frac{d\sigma'}{\sqrt{\Psi(\nu)+C}} = 0 \ldots\ldots\ldots\ldots (3).$$

3. The lines of curvature are obviously included in the above differential equation, hence if v^2 is of the form

$$\frac{\Phi(\mu)+\Psi(\nu)}{\mu^2-\nu^2},$$

the particle can, by a proper determination of the initial circumstances, be made to describe a line of curvature; if then it be required to make the particle, under the action of a force which causes the potential, and consequently v^2, to assume the above form, describe the line of curvature $\mu = \mu'$, it must obviously be projected in the direction of a tangent to this curve, and it remains to find the proper initial velocity β.

Let N be the resolved part of the force along a line perpendicular to the direction of motion and the normal to the surface, ρ the radius of curvature of the orbit, and ϕ the angle the osculating plane makes with the tangent plane to the surface; we have then the following equation from mechanical considerations :—

$$N + \frac{v^2}{\rho}\cos\phi = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If, now, the path is the line of curvature $\mu = \mu'$, $\frac{\cos\varphi}{\rho} = \frac{1}{R'}$ when R' is the radius of curvature of the section of the hyperboloid of one sheet by the tangent plane to the ellipsoid. Now, $R' = \frac{\mu^2-\nu^2}{p'}$, p' being the perpendicular from the centre on the tangent plane to the

hyperboloid of one sheet, hence (1) becomes

$$N+\frac{v^2}{\mu^2-\nu^2}p'=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now $\frac{1}{2}\frac{dv^2}{d\sigma}$ is evidently the force perpendicular to the μ line of curvature or N; therefore, differentiating the equation

$$v^2(\mu^2-\nu^2)=\Phi(\mu)+\Psi(\nu),$$

we get

$$(\mu^2-\nu^2)\frac{dv^2}{d\sigma}+v^2\times 2\mu\,.\,\frac{d\mu}{d\sigma}=\Phi'(\mu)\frac{d\mu}{d\sigma};$$

or

$$\tfrac{1}{2}\frac{dv^2}{d\sigma}+\frac{v^2\mu\frac{d\mu}{d\sigma}}{\mu^2-\nu^2}=\tfrac{1}{2}\,.\,\frac{\Phi'(\mu)\frac{d\mu}{d\sigma}}{\mu^2-\nu^2}.$$

But $\frac{\mu d\mu}{d\sigma}=p'$, therefore

$$N+\frac{v^2p'}{\mu^2-\nu^2}=\tfrac{1}{2}\,.\,\frac{\Phi'(\mu)\frac{d\mu}{d\sigma}}{\mu^2-\nu^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and this will be identical with (2) if $\Phi'(\mu)=0$. This equation will determine β, since $\Phi(\mu)$ involves μ', ν', and β. The following examples will make this more clear.

A particle constrained to move on the surface of an ellipsoid is acted on by an attractive force to the centre as Ar. The potential is then $\frac{1}{2}Ar^2$, and consequently $v^2=A(r'^2-r^2)+\beta^2$ or $A(\mu'^2+\nu'^2-\mu^2-\nu^2)+\beta^2$, since $r^2=\mu^2+\nu^2+a^2-h^2-k^2$ (Salmon's *Surfaces*, Art. 161);

hence

$$v^2=\frac{\{A(\mu'^2+\nu'^2)+\beta^2\}(\mu^2-\nu^2)-A(\mu^4-\nu^4)}{\mu^2-\nu^2},$$

which gives

$$(\mu)\equiv\{A(\mu'^2+\nu'^2)+\beta^2\}\mu^2-A\mu^4,$$

and the condition $\Psi'(\mu)=0$ determines β from the equation

$$A(\mu'^2+\nu'^2)+\beta^2-2A\mu'^2=0,$$

or

$$\beta^2=A(\mu^2-\nu'^2),$$

and therefore

$$v^2=A(\mu'^2-\nu^2).$$

A particle moves on the surface, and is acted on by a central force situated on the axis major at a point S distant $\frac{hk}{a}$ from the centre, the force varying directly as the distance and inversely as the cube of the tangent to the sphere having S for centre and touching the ellipsoid at the umbilics, to determine the initial velocity in order that it may describe the line of curvature $\mu=\mu'$. The radius of this sphere will be found to be $\frac{bc}{a}$, and the tangent from any point on the ellipsoid will be consequently equal in length to $\mu+\nu$. Now the potential is $\frac{A}{\mu+\nu}$,

hence $$\tfrac{1}{2}v^2 = \frac{A}{\mu+\nu} - \frac{A}{\mu'+\nu'} + \tfrac{1}{2}\beta^2,$$

or $$v^2 = \frac{(\mu^2-\nu^2)\left(\beta^2 - \dfrac{2A}{\mu'+\nu'}\right) + 2A(\mu-\nu)}{\mu^2-\nu^2};$$

hence $$\Phi(\mu) = \left(\beta^2 - \frac{2A}{\mu'+\nu'}\right)\mu^2 + 2A\mu;$$

and the condition $\Phi(\mu') = 0$ gives

$$\beta^2 = A\left(\frac{\mu'-\nu'}{\mu'+\nu'}\right).$$

Substituting this value of β^2, we find, since $r = \mu'+\nu$,

$$\frac{v^2}{A} = \frac{2}{r} - \frac{1}{\mu'}.$$

A particle moves on the surface and is acted on by an attractive force situated at the centre, S, of the sphere having double contact at the umbilics whose abscissa is positive, and by a repulsive force situated at S', the centre of the sphere having double contact at the umbilics whose abscissa is negative, the law of force being, as in the preceding problem, to determine the initial velocity, in order that the path may be the line of curvature $\mu = \mu'$.

The principle of vis viva gives us

$$v^2 = \frac{2A}{\mu+\nu} - \frac{2B}{\mu-\nu} + C,$$

where A and B are the absolute forces respectively,

and $$C = \beta^2 - \frac{2A}{\mu'+\nu'} + \frac{2B}{\mu'-\nu'}.$$

The condition $\Phi'(\mu) = 0$ gives us

$$\beta^2 = \frac{A(\mu'-\nu')}{\mu'+\nu'} - \frac{B(\mu'+\nu')}{\mu'-\nu'}.$$

If now we make $\beta^2 = 0$, we get

$$\mu' = \left(\frac{\sqrt{A}+\sqrt{B}}{\sqrt{A}-\sqrt{B}}\right)\nu' = a\nu' \text{ say.}$$

Hence we see that, if the particle be placed at rest at any point of the curve $\mu = a\nu'$, it will describe the line of curvature $\mu = \mu'$ which passes through the point, and will obviously oscillate perpetually.

The curve $\mu = a\nu$ is the intersection of a sphere with the surface.

4. I now show how to find the reaction R of the surface for any cases of motion on the surface in which the forces have a potential.

Now $\frac{1}{2}\frac{dv^2}{d\sigma_0}$ is evidently the resolved part of the force perpendicular to the surface, $d\sigma_0$ being the element of the normal, and consequently

$$R = \tfrac{1}{2}\frac{dv^2}{d\sigma_0} + \frac{v^2}{\rho}\cos\theta,$$

θ being the angle the osculating plane makes with the normal. But $\frac{\cos\theta}{\rho} = \frac{1}{\rho_0}$, where ρ_0 is the radius of curvature of the normal section through the tangent line; and again $\rho_0 = \frac{\gamma^2}{p}$ (Salmon's *Geometry of Three Dimensions*, Art. 194), where γ is the semi-diameter parallel to the tangent line and p the perpendicular on the tangent plane from the centre; hence

$$R = \tfrac{1}{2}\frac{dv^2}{d\sigma_0} + \frac{v^2 p}{\gamma^2}.$$

But v^2 is a function of μ, ν, and a the primary axis of the surface, and if we consider for a moment a as variable where it enters through x, y, and z,

$$p d\sigma_0 = a da,$$

therefore finally

$$R = \tfrac{1}{2}\frac{p}{a}\cdot\frac{dv^2}{da} + \frac{pv^2}{\gamma^2} \quad \text{.....(1).}$$

5. I now discuss some special cases of motion by assigning particular forms to $\Phi(\mu)$ and $\Psi(\nu)$ which render $\Phi(\mu)+\Psi(\nu)$ divisible by $\mu^2-\nu^2$.

In general, the differential equation of the path is

$$\frac{d\sigma}{\sqrt{\Phi(\mu)-C}} \pm \frac{d\sigma'}{\sqrt{\Psi(\nu)+C}} = 0.$$

Let us make

$$\Phi(\mu) - C = \frac{B}{\mu^2-h^2},$$

$$\Psi(\nu) + C = \frac{B}{h^2-\nu^2},$$

where B is a constant. The differential equation of the path then assumes the form

$$\sqrt{\frac{a^2-\mu^2}{k^2-\mu^2}}\,d\mu \pm \sqrt{\frac{a^2-\nu^2}{k^2-\nu^2}}\cdot d\nu = 0,$$

which is the differential equation of the orthogonal trajectory to an umbilicar geodesic. We have also

$$v^2 = \frac{\Phi(\mu)+\Psi(\nu)}{\mu^2-\nu^2} = \frac{B}{(\mu^2-h^2)(h^2-\nu^2)}.$$

Now

$$y^2 = \frac{(a^2-h^2)(\mu^2-h^2)(h^2-\nu^2)}{h^2(k-h^2)},$$

hence $v^2 = \frac{A}{y^2}$, A being constant, and consequently the force is attractive and varies as $\frac{A}{y^3}$.

The principle of vis viva gives us

$$v^2 = \frac{A}{y^2} - \frac{A}{y'^2} + \beta^2,$$

therefore

$$\beta^2 = \frac{A}{y'^2}.$$

Hence:—*A particle constrained to move on the surface under the action of an attractive force varying as* $\frac{A}{y^3}$, *and projected perpendicularly to an umbilicar geodesic with the initial velocity* $\frac{\sqrt{A}}{y'}$, *will describe the curve traced on the surface by the extremity of an umbilicar geodesic of constant length.*

On the Points and Tangents common to Two Conics.

By Professor Genese, M.A.

[*Read May 10th*, 1883.]

Taking the triangle ABC formed by three common tangents to two conics as that of reference in any system of point coordinates, let the equations to the conics be

$$\sqrt{la} + \sqrt{m\beta} + \sqrt{n\gamma} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\sqrt{l'a} + \sqrt{m'\beta} + \sqrt{n'\gamma} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

each radical being capable of the double sign; we may, without loss of generality, make the convention that the first term in each is to be taken positively.

For the points common to the two conics, we have

$$\frac{\sqrt{a}}{\sqrt{m}\sqrt{n'} - \sqrt{n}\sqrt{m'}} = \frac{\sqrt{\beta}}{\sqrt{n}\sqrt{l'} - \sqrt{l}\sqrt{n'}} = \&c.,$$

or

$$\frac{a}{mn' + nm' - 2\sqrt{m}\sqrt{n}\sqrt{m'}\sqrt{n'}} = \&c.,$$

or, say,

$$\frac{a}{p - (p')} = \frac{\beta}{q - (q')} = \frac{\gamma}{r - (r')} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where p', q', r' may take the double sign; but, since $q'r' = ll'p'$ always,

the product $p'q'r'$ must be positive. Thus p', q', r' being certain numerical quantities, the coordinates of the points P, Q, R, S common to the two conics are

$$(p-p', q-q', r-r') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots P,$$
$$(p-p', q+q', r+r') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots Q,$$
$$(p+p', q-q', r+r') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots R,$$
$$(p+p', q+q', r-r') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots S.$$

The equation to AP is $\dfrac{\beta}{q-q'} = \dfrac{\gamma}{r-r'}$;

to BS is $\dfrac{\gamma}{r-r'} = \dfrac{\alpha}{p+p'}$;

to CR is $\dfrac{\alpha}{p+p'} = \dfrac{\beta}{q-q'}$;

thus these straight lines intersect at the point

$$(p+p', q-q', r-r'),$$

that is, the triangles ABC, PSR are homologous. In the same way, it may be shown that the triangle ABC is homologous with each of the triangles QRS, RQP, and SPQ (the positions of the letters indicating the corresponding points).

The four centres of homology are the points

$$(p+p', q-q', r-r'),$$
$$(p+p', q+q', r+r'),$$
$$(p-p', q-q', r+r'),$$
$$(p-p', q+q', r-r').$$

It follows, from the above, that *each of the triangles determined by the common tangents to two conics is homologous with each of the triangles whose vertices are at the points common to the conics.*

The theorem may be restated thus :—*If two triangles be such that more than one conic can be simultaneously described about one and inscribed in the other, then the two triangles are homologous.*

A direct analytical proof of this theorem seems to be exceedingly difficult.

It may be added, that the equations to the axes of homology take the forms

$$\frac{\alpha}{p'} \pm \frac{\beta}{q'} \pm \frac{\gamma}{r'} = 0.$$

If the conics be traced, the points P, Q, R, S may be determined by the aid of the following lemma. Let L, M, N be the points of contact of the conic (1) with BC, CA, AB respectively. Then, whatever be the nature of the conic, for the part of the curve from M to N not passing through L (including the case of a passage through a point at

infinity) the signs of the radicals in (1) are given by

$$\sqrt{m\beta} + \sqrt{n\gamma} - \sqrt{l\alpha} = 0.$$

From N to L the equation is

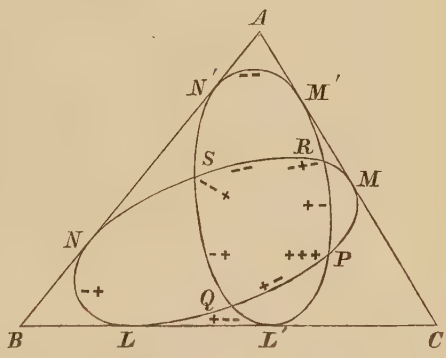
$$\sqrt{n\gamma} + \sqrt{l\alpha} - \sqrt{m\beta} = 0;$$

and from L to M $\quad \sqrt{l\alpha} + \sqrt{m\beta} - \sqrt{n\gamma} = 0.$

Hence, since it is convenient to give the positive sign to $\sqrt{l\alpha}$, the signs to be taken for $\sqrt{m\beta}$, $\sqrt{n\gamma}$ are—

from M to N, $- -$;
from N to L, $- +$;
from L to M $+ -$.

Treating equation (2) similarly, there is no difficulty in determining the proper signs to be given to p', q', r' in (3) for a particular point of intersection of (1) and (2). The figure shows the determination in the case of two ellipses.



Two signs, as $- -$, are according to the lemma.

Three signs, as $+ - -$, are those to be given in (3) to p', q', r' respectively; they are to be changed on clearing the brackets.

The theorems admit of easy geometrical verification on projecting the quadrilateral $PQSR$ into a square.

Thursday, June 14th, 1883.

Prof. HENRICI, F.R.S., President, in the Chair.

Prof. W. Woolsey Johnson, of Annapolis, was admitted into the Society. Prof. Cayley made a few remarks "On Seminvariants and Mr. Hammond's recent Discovery." Prof. Sylvester (who was warmly received) and Mr. Hammond spoke on the same subject. Mr. Tucker read parts of papers: by Prof. H. Lamb, "On the Mutual Potential of

Two Lines in Space"; by Mr. H. M. Jeffery, "On Bicircular Quartics with Collinear Foci"; and spoke on the subject of "Inverse Coordinate Curves" (*i.e.*, such that $xx' = a^2 = yy'$).

The following presents were received:—

"Educational Times," for June, 1883.

"Cambridge Philosophical Society—Proceedings," Vol. iv., Parts 2, 3, 4, and 5; "Transactions," Vol. xiii., Part 2.

"Proceedings of the Physical Society of London," Vol. v., Part 3.

"Great Trigonometrical Survey of India," Vols. vii. and viii.

"Johns Hopkins University Circulars," Vol. ii., No. 22.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," 1883, Parts 1 to 21; 8vo, Berlin, 1883.

"Atti della R. Accademia dei Lincei," Vol. vii., Fasc. 9 and 10.

"Journal de l'École Polytechnique, &c.," 52 Cahier.

"Bulletin des Sciences Mathématiques, &c.," Tome vi., Index.

"Bulletin Astronomique et Météorologique de l'Observatoire de Rio de Janeiro," No. 3.

"Grundzüge einer Arithmetischen Theorie der Algebraischen Grössen," Festschrift zu Herrn Ernst Eduard Kummer's Fünfzigjährigem Doctor-Jubiläum, 10 Sept., 1881, von L. Kronecker; 4to, Berlin, 1882.

"Zur Theorie der Abel'schen Gleichungen," von Kronecker; 4to.

"Sur les Unités Complexes," par M. L. Kronecker; 4to, Paris, 1883.

The following pamphlets by M. Kronecker:—

"Ueber Potentiale n-facher Mannigfaltigkeiten."

"Die Composition Abel'scher Gleichungen."

"Die Kubischen Abel'schen Gleichungen des Bereichs $(\sqrt{-31})$."

"Ueber den vierten Gauss'schen Beweis des Reciprocitätsgesetzes für die Quadratischen Reste."

"Ueber die Irreductibilität von Gleichungen."

"Ueber die Symmetrischen Functionen."

"Zur Theorie der Elimination einer Variabeln aus zwei Algebraischen Gleichungen."

"Zur Theorie der Elliptischen Functionen."

"Aus der Theorie der Algebraischen Gleichungen."

"Beiblätter zu den Annalen der Physik und Chemie," Band vii., St. 4 and 5.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," Tome v., 2 Cahier.

"Jornal de Sciencias Mathematicas e Astronomicas," publicado pelo Dr. Francisco Gomes Teixeira, Vol. iv., Nos. 1—6; 8vo, Coimbra, 1882.

"Atti del Reale Istituto Veneto di Scienze, Lettere, ed Arti," Tomo Primo, Serie Sesta, Dispensa 1, 2, 3, Tomo Ottavo; Serie Quinta, Dispensa 7, 8, 9, 10; 8vo, Venice.

On Bicircular Quartics with Collinear Foci.

By Henry M. Jeffery.

[*Read June 14th*, 1883.]

1. Dr. Casey, in his classical treatise "On Bicircular Quartics" (Dublin, 1869), has omitted to treat of these degenerate forms, when the foci are collinear, with the single exception of Cartesians. Mr. Hart, to whom we are indebted for the complete analytical exposition of the theory by which bicirculars are generated (*Proceedings*, Vol. xi., p. 143), has also passed this group in silence.

Having had occasion to study this group of quartics in the dual form, I have proposed to obtain its equation (1) by Dr. Casey's method of generation, (2) in the vector form, and otherwise, in terms of the focal distances from a double focus.

2. It may be stated *in limine* that the group of bicirculars is degenerate, inasmuch as they may be generated by three, and not four, sets of orthotomic circles, and that all these three circles of reciprocation cannot be referred to the same self-conjugate triangles with the corresponding focal conics. See §§ 4, 9.

The several varieties of these quartics caused by the omission of terms in their equation will be considered. The variety, in which one of the single foci coincides with a double, and forms a triple focus, has been already discussed (*Proceedings*, Vol. xii., pp. 17—27). A separate section will be assigned to Cartesians, in which Dr. Casey's results will be obtained in a different way.

All singular forms will be exhibited, and a method given for determining the inflexions in these ovals.

3. Bicircular quartics with collinear foci may be expressed in either of these forms:

$$h(x^2+y^2)^2+(1+kx)(x^2+y^2)+u_1=0,$$

or

$$h(x^2+y^2)^2+x^2+y^2+u_2=0.$$

When the origin is arbitrary, these quartics have the fuller form

$$h(x^2+y^2)^2+(1+kx)(x^2+y^2)+mx^2+nx+p=0.$$

The double foci are found in the usual way by the intersection of two hyperbolæ

$$4h(x^2-y^2)+2kx+m=0,$$

$$4hxy+ky=0.$$

They lie on the (x) axis

$$4hx^2+2kx+m=0.$$

If the origin be transposed to either double focus, the quartic takes the first form; if mid-way between them, the second form. The first form is used in this memoir, with the parameters κ, λ,

$$\lambda(x^2+y^2)^2+(1+fx)(1+fx)(x^2+y^2)=\kappa(1+ex).$$

If used to denote a group of class-quartics, this form would represent all those which have a triple and a single focus, and one of its satellite-foci coincident with the triple focus.

4. To generate a bicircular quartic with collinear foci by Dr. Casey's method of reciprocation.

There are in this case three focal conics, which are doubly confocal, since their common foci are the double foci of the quartic. (Casey, p. 469.)

$$\lambda(x^2+y^2)-(x-a)^2=0 \ldots\ldots (F_1),$$
$$\mu(x^2+y^2)-(x-b)^2=0 \ldots\ldots (F_2),$$
$$\nu(x^2+y^2)-(x-c)^2=0 \ldots\ldots (F_3).$$

And there are three corresponding circles of reciprocation, which are mutually orthotomic (Casey, p. 469), and whose centres are collinear.

$$(x-f)^2+y^2=\rho_1^2 \ldots\ldots (J_1),$$
$$(x-g)^2+y^2=\rho_2^2 \ldots\ldots (J_2),$$
$$(x-h)^2+y^2=\rho_3^2 \ldots\ldots (J_3).$$

Transform F_1 and J_1 to the centre of J_1, and write r^2 for x^2+y^2,

$$\lambda(x+f)^2+\lambda y^2-(x+f-a)^2=0 \ldots\ldots (F_1).$$

Its Boothian equivalent tangential equation is

$$\lambda(1+f\xi)^2-\{1+(f-a)\xi\}^2-\eta^2=0 \ldots\ldots (F_1).$$

By reciprocating from the centre of J_1 by the relation

$$\frac{1}{p}=\frac{2r}{r^2+\rho_1^2}, \text{ where } \frac{1}{p^2}=\xi^2+\eta^2,$$

the bicircular is generated

$$\lambda(r^2+2fx+\rho_1^2)^2-\{r^2+2(f-a)x+\rho_1^2\}^2-4a^2y^2=0.$$

Again, transform to the common focus as origin:

$$\lambda(r^2+\rho_1^2-f^2)^2-(r^2-2ax+\rho_1^2-f^2+2af)^2-4a^2y^2=0.$$

Since the focal conics are confocal,

$$\frac{2a}{\lambda-1}=\frac{2b}{\mu-1}=\frac{2c}{\nu-1}=d,$$

if d denote the distance between their common foci.

The quartic may now be written

$$r^4+2dxr^2+2r^2(\rho_1^2-f^2-df-ad)+2dx(\rho_1^2-f^2+2af)$$
$$+(\rho_1^2-f^2-df)^2-f^2(d^2+2ad)=0.$$

Two other forms of the equation may be written, in which ρ_2, g, b; ρ_3, h, c; take the places of ρ_1, f, a. These forms are identified by equating their coefficients; so that

$$(2f+d)(2a+d)=(2g+d)(2b+d)=(2h+d)(2c+d)=S$$
$$=\surd\{d(d+2a)(d+2b)(d+2c)\}.$$

Since $4\rho_1^2-(2f+d)^2+d^2-4ad=-\dfrac{S^2}{(d+2a)(d+2b)}+d^2-2ad-2bd,$

by symmetry it also $=-\dfrac{S^2}{(d+2b)(d+2c)}+d^2-2bd-2cd;$

whence S is known.

For $$\rho_1^2=\frac{b-a}{4b+2d}\{(2f+d)^2-d(2b+d)\}=\frac{d(a-b)(a-c)}{2a+d},$$

$$\rho_2^2=\frac{d(b-a)(b-c)}{2b+d};\quad \rho_3^2=\frac{d(c-a)(c-b)}{2c+d}.$$

Hence it appears that the conic F_2, which corresponds to the imaginary circle J_2, always lies between the two conics F_1 and F_3 (Mr. Hart, *Proceedings*, Vol. xi., p. 149). a, b, c are supposed to be in order of magnitude.

The circles of reciprocation are seen to be orthotomic.

For $$\rho_2^2+\rho_3^2=\frac{d(d+2a)(b-c)^2}{(d+2b)(d+2c)}=(g-h)^2;$$

$$\rho_3^2+\rho_1^2=(h-f)^2;\quad \rho_1^2+\rho_2^2=(f-g)^2.$$

Finally, the constants in the bicircular equation may be expressed in terms of a, b, c, the given distances of the directrices and d, that between the foci

$$r^4+2dxr^2-(a+b+c)\,dr^2+dx(S-d^2-ad-bd-cd)$$
$$+\frac{d^2}{4}\{a^2+b^2+c^2-2bc-2ca-2ab-2(a+b+c+d)\,d+2S\}=0;$$

where $S=\surd\{d(d+2a)(d+2b)(d+2c)\}$, and $r^2=x^2+y^2$.

Cor.— Conversely, by this form of the equation, a fundamental theorem of Dr. Casey's is illustrated; the foci of the confocal focal conics are double foci of the bicircular quartic.

5. The bicircular quartic is also generated in three ways as the envelope of a variable circle, whose centre lies on (F), a focal conic, and which cuts (J), the corresponding circle of reciprocation, orthogonally. (Casey, p. 469.)

Let the generating circle be

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2,$$

where the parameters are subject to two conditions :

$$\lambda(\alpha^2+\beta^2)=(a-\alpha)^2,$$

and $$2af-\alpha^2-\beta^2+\gamma^2-\rho_1^2-f^2=0;$$

because it cuts orthogonally the circle

$$(x-f)^2+y^2=\rho_1^2.$$

The resulting quartic is found to be identical with the preceding

$$\lambda(r^2+\rho_1^2-f^2)^2-(r^2-2ax+\rho_1^2-f^2+2af)^2-4a^2y^2=0.$$

Hence it appears that the form of reciprocation

$$\frac{1}{p}=\frac{2r}{r^2+\rho}$$

expresses both the properties stated in the enunciation.

The dual theorem may be obtained from this genesis. Class-quartics which have a triple and single focus, and whose satellite conic has one of its collinear foci coincident with the triple focus, as denoted by the Boothian equation

$$\kappa(1+e\xi)=(1+f\xi)(\xi^2+\eta^2)+\lambda(\xi^2+\eta^2)^2,$$

are generated in three ways. They are envelopes of variable singly confocal conics, whose directrix touches F a dirigent circle, and whose common tangent with J subtends a right angle at the common focus.

6. In the special case, when there is a triple and a double focus, the coefficient of x vanishes,

and $$S=d(d+a+b+c);$$

whence $$d(a^2+b^2+c^2-2bc-2ca-2ab)=8abc,$$

and d is thus a function of a, b, c.

The equation to the bicircular becomes

$$\frac{1}{d}(x^2+y^2)^2+(2x-a-b-c)(x^2+y^2)+2abc=0.$$

Let this form be compared with the equation (*Proceedings*, Vol. xii., p. 18) in which the constants are functions of the focal distances,

$$\Sigma(a'^2-2b'c')(x^2+y^2)^2-2a'b'c'(a'+b'+c'+4x)(x^2+y^2)+a'^2b'^2c'^2=0.$$

It follows that $$a:b:c:1::a':b':c':2.$$

The distances of the single foci from the triple focus are therefore double the distances of the directrices of the focal conics from the same triple focus.

7. In the case, when the bicircular passes through a double focus,

the constant term disappears, d is a function of a, b, c, and the quartic becomes

$$r^4+2dxr^2-(a+b+c)\,dr^2-\tfrac{1}{2}dx\,(a^2+b^2+c^2-2bc-2ca-2ab)=0,$$

where $$4d\Sigma\,[a\,(a^2-b^2-c^2)]=\{\Sigma\,(a^2-2bc)\}^2.$$

8. If the focal conics be parabolæ, since d is infinite,

$$a+f=b+g=c+h=\tfrac{1}{2}\,(a+b+c)=s\ \text{(suppose)},$$

$$\rho_1^2=(a-b)(a-c),\ \rho_2^2=(b-a)(b-c),\ \rho_3^2=(c-a)(c-b).$$

The bicircular quartic degenerates into the circular cubic with four single foci collinear with its double focus:

$$(x-s)(x^2+y^2)-\frac{x}{4}(a^2+b^2+c^2-2bc-2ca-2ab)-(s-a)(s-b)(s-c)=0;$$

where a, b, c denote the distances of the directrices of the focal conics from the double focus.

9. Let the case be considered where the focal conics are denoted, as before:

$$\lambda\,(x^2+y^2)-(x-a)^2=0\ \ldots\ldots\ldots\ldots\ldots\ldots(F_1),$$

$$\mu\,(x^2+y^2)-(x-b)^2=0\ \ldots\ldots\ldots\ldots\ldots\ldots(F_2);$$

but the orthotomic circles of reciprocation J_1, J_2 are referred to the same self-conjugate triangles

$$x^2+y^2-2bx+ab=0\ldots\ldots\ldots\ldots\ldots\ldots(J_1),$$

$$x^2+y^2-2ax+ab=0\ldots\ldots\ldots\ldots\ldots\ldots(J_2).$$

The bicircular quartic, generated from either pair, is

$$(x^2+y^2)^2+2\,(dx-ab-ad-bd)(x^2+y^2)+2abdx+a^2b^2=0.$$

The third pair F_3 and J_3, from which the same quartic may be generated, are not, as in the general theorem, referred to the same self-conjugate triangle.

10. To determine the vector equation; *i.e.*, to obtain the linear relation between the distances from three of its single foci of a point in a bicircular quartic whose foci are collinear.

Let the two forms of the quartic equation be compared, in the first of which the origin is a double focus.

$$\lambda\,(x^2+y^2)^2+(1+fx)(x^2+y^2)=\kappa\,(1+ex),$$

$$lr_1+mr_2+nr_3=0,$$

where $r_1^2=(x+a)^2+y^2$, and r_2, r_3 have similar values; a, b, c are now the distances of three foci from the double focus. Then, on expanding, it will be found that

$$l\sqrt{a}+m\sqrt{b}+n\sqrt{c}=0,$$

since there is no coefficient of x^2, other than that of x^2+y^2. Let d

denote the fourth focal distance; the fourth focus may be expressed as a point-circle in either of the forms

$$(x+d)^2+y^2=0,$$
$$\lambda r_1^2+\mu r_2^2+\nu r_3^2=0.$$

Since this point-circle touches the quartic $(lr_1+mr_2+nr_3=0)$,

$$\frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{n^2}{\nu}=0.$$

The discriminant of this circle denotes that it is the intersection of two imaginary straight lines. This condition of linear factors is

$$\frac{(b-c)^2}{\lambda}+\frac{(c-a)^2}{\mu}+\frac{(a-b)^2}{\nu}=0.$$

Equate coefficients in the two forms of the point-circle

$$\lambda a^2+\mu b^2+\nu c^2=(\lambda+\mu+\nu)\,d^2,$$
$$\lambda a+\mu b+\nu c=(\lambda+\mu+\nu)\,d;$$

whence $\lambda(a-d):\mu(b-d):\nu(c-d)::b-c:c-a:a-b,$

i.e. $\frac{1}{\lambda}:\frac{1}{\mu}:\frac{1}{\nu}::AB.AC.AD:BA.BC.BD:CA.CB.CD$

(Salmon's *Higher Plane Curves*, p. 237). The relative values of l, m, n are found from the relations

$$l\sqrt{a}+m\sqrt{b}+n\sqrt{c}=0,$$
$$\frac{l^2(a-d)}{b-c}+\frac{m^2(b-d)}{c-a}+\frac{n^2(c-d)}{a-b}=0.$$

The linear relation required is found to be

$$r_1(b-c)\{\sqrt{(ad)}-\sqrt{(bc)}\}$$
$$+r_2(c-a)\{\sqrt{(bd)}-\sqrt{(ca)}\}+r_3(a-b)\{\sqrt{(cd)}-\sqrt{(ab)}\}=0.$$

By forming three other linear relations by symmetry, and adding all the four, we obtain the most general vector equation

$$Ar_1+Br_2+Cr_3+Dr_4=0.$$

Cor. 1.—If $d=\infty$, the vector equation to Cartesians is, as in § 28,

$$(b-c)\sqrt{a}\,r_1+(c-a)\sqrt{b}\,r_2+(a-b)\sqrt{c}\,r_3=0.$$

Cor. 2.—If $a^{-\frac{1}{2}}+b^{-\frac{1}{2}}+c^{-\frac{1}{2}}+d^{-\frac{1}{2}}=0$, and the constant term vanishes; the bicircular passes through a double focus (§ 11), and its vector equation is $\frac{\sqrt{b}-\sqrt{c}}{\sqrt{a}}r_1+\frac{\sqrt{c}-\sqrt{a}}{\sqrt{b}}r_2+\frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}}r_3=0.$

The result may be simply verified by comparing the forms

$$\lambda(x^2+y^2)^2+(1+fx)(x^2+y^2)=\kappa ex,$$
$$lr_1+mr_2+nr_3=0.$$

The conditions are

$$la+mb+nc=0,$$
$$l\sqrt{a}+m\sqrt{b}+n\sqrt{c}=0.$$

Cor. 3.—If $d = 0$, or the bicircular have a triple and a double focus, the vector equation is

$$\frac{b-c}{\sqrt{a}} r_1 + \frac{c-a}{\sqrt{b}} r_2 + \frac{a-b}{\sqrt{c}} r_3 = 0.$$

This form is otherwise obtained (Vol. xii., p. 18).

Cor. 4.—If the term involving x^2+y^2 be removed, no vector equation is possible.

The values of l, m, n cannot be real, since the conditions

$$l\sqrt{a} + m\sqrt{b} + n\sqrt{c} = 0,$$
$$\Sigma l^4 a^2 - \Sigma m^2 n^2 (b^2+c^2) = 0,$$

when combined, lead to the sum of squares

$$\frac{1}{l^2}(b-c)^2 + \frac{1}{m^2}(c-a)^2 + \frac{1}{n^2}(a-b)^2 = 0.$$

In class-quartics, this group has a satellite-focus at infinity.

Cor. 5.—If an apse be at infinity, $\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d} = 0$, the bicircular degenerates into the circular cubic with collinear foci,

$$r_1(\sqrt{b}-\sqrt{c}) + r_2(\sqrt{c}-\sqrt{a}) + r_3(\sqrt{a}-\sqrt{b}) = 0.$$

11. To determine the focal distances of the apses in terms of the distances of the four single foci from a double focus.

At an apse, $y = 0$, r_1, r_2, r_3 become $x+a$, $x+b$, $x+c$ severally. The vector equation of § 10 becomes

$$\Sigma (x+a)(b-c)\{\sqrt{(ad)} - \sqrt{(bc)}\} = 0.$$

After rejecting the factor

$$(\sqrt{b}-\sqrt{c})(\sqrt{c}-\sqrt{a})(\sqrt{a}-\sqrt{b}),$$

$$x(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}) + \sqrt{(bcd)} + \sqrt{(acd)} + \sqrt{(abd)} + \sqrt{(abc)} = 0.$$

The other three apses are determined, by altering the signs of the radicals by pairs. Thus

$$x(\sqrt{a}+\sqrt{b}-\sqrt{c}-\sqrt{d}) + \sqrt{(abcd)}\left\{\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} - \frac{1}{\sqrt{c}} - \frac{1}{\sqrt{d}}\right\} = 0.$$

Cor. 1.—For Cartesians, when $d = \infty$, the distances of the apses are thus obtained:

$$x = \sqrt{(bc)} + \sqrt{(ca)} + \sqrt{(ab)}.$$

Cor. 2.—If the origin be an apse, so that

$$a^{-\frac{1}{2}} + b^{-\frac{1}{2}} + c^{-\frac{1}{2}} + d^{-\frac{1}{2}} = 0,$$

the distances of the other three apses have the type

$$x\{(\sqrt{a}+\sqrt{b})(\sqrt{a}+\sqrt{c}) - 2a\} = 2a\sqrt{(bc)}.$$

Cor. 3.—If $d = 0$, for the apses

$$x(\sqrt{a}+\sqrt{b}+\sqrt{c}) = \sqrt{(abc)}.$$

Cor. 4.—If an apse be at infinity, the distances of the three other apses from the double focus of the circular cubic have the type

$$\tfrac{1}{2}\sqrt{a}(\sqrt{a}+\sqrt{b}+\sqrt{c})-\tfrac{1}{2}\sqrt{(bc)}.$$

12. If the bicircular cubic be nodal, as for instance the pedal curve of a central conic, two foci, as well as two apses, unite at the node and disappear.* (Casey, p. 474.)

Its vector equation is reduced, when $c = d$, by rejecting the factor $\sqrt{a}-\sqrt{b}$,

$$(r_1 b+r_2 a)\sqrt{c}-(r_1+r_2)c\sqrt{c}+r_3(\sqrt{a}+\sqrt{b})\{c-\sqrt{(ab)}\} = 0.$$

13. To determine the collinear foci in these bicircular quartics.

The equation to the group being

$$\lambda(x^2+y^2)^2+(1+fx)(x^2+y^2) = \kappa(1+ex),$$

it may be proved that there are two double foci, at the intersection of the quadrics

$$2\lambda(x^2-y^2)+fx = 0,$$

$$4\lambda xy+fy = 0.$$

These are the origin $(0, 0)$, and the point $\left(-\frac{f}{2\lambda}, 0\right)$.

These two double foci and the four single foci may be obtained from the equivalent class-equation

$$S^3-27T^2 = 0,$$

where S, T denote the quartic and sextic invariants.

$$3S = Q^2-\tfrac{3}{4}RV-3\kappa\lambda R^2,$$

$$27T = -Q^3+\tfrac{9}{8}QRV+9\kappa WR^2,$$

and

$$Q = 2\lambda+\xi^2+\eta^2+f\xi,$$

$$R = \xi^2+\eta^2,$$

$$V = (\xi+f)^2+\eta^2+4\kappa\lambda e\xi-\kappa ef\eta^2,$$

$$W = \lambda^2+\tfrac{1}{2}\lambda(\xi^2+\eta^2+f\xi)-\tfrac{3}{16}\eta^2(f^2-ef-e^2\kappa\lambda).$$

Neglecting for the time the third terms in both S and T, which involve κ, we find from the expansion of $(3S)^3-(27T)^2$,

$$\tfrac{27}{64}R^2V^2(Q^2-RV).$$

Hence R^2 or $(\xi^2+\eta^2)^2$ measures the expansion, and the class of te

* It is incorrectly stated (*Proceedings*, Vol. xiii., p. 71) that this node is a double focus. The equivalent class-curve is a sextic, with two double and two single foci.

equivalent equation is the eighth, as was known, since there are two double points at infinity.

Reject R^2, and also suppress R or $\xi^2+\eta^2$, where it occurs, to obtain the foci. There results

$$\tfrac{2}{6}\tfrac{7}{4}\,(f^2+2f\xi+4\kappa\lambda e\xi-\kappa ef\eta^2)^2\,(4\lambda^3+4\lambda f\xi-f^2\eta^2)$$
$$-\tfrac{2}{1}\tfrac{7}{6}\kappa\eta^2\,(2\lambda+f\xi)^3\,(f^2-fe-e^2\kappa\lambda).$$

Write $(\xi^2+\eta^2)-\xi^2$ for η^2, and again reject $\xi^2+\eta^2$. Then $(2\lambda+f\xi)^2$ is a common factor, and denotes a double focus.

The four single foci are thus expressed:

$$(f^2+2f\xi+4\kappa\lambda e\xi+\kappa ef\xi^2)^2+8\kappa\xi^2\,(f^2-ef-e^2\kappa\lambda)\,(2\lambda+f\xi),$$

or $f^2+4\,(f+2\kappa\lambda e)\,\xi+(4+2\kappa ef+16\kappa\lambda)\,\xi^2+4\kappa\,(2f-e)\,\xi^3+\kappa^2e^2\xi^4=0.$

The origin is another double focus, since the class is the eighth.

14. To express the equation to the group of bicircular quartics in terms of the distances of the single foci from a double focus.

Let the binary quartic of § 13 be compared with the form which denotes the product of these focal distances,

$$(1+a\xi)\,(1+b\xi)\,(1+c\xi)\,(1+d\xi)=0,$$

or
$$1+p\xi+q\xi^2+r\xi^3+s\xi^4=0,$$

where p, q, r, s mean the usual combinations of the roots. The following relations may be established:

$$\lambda : f : \kappa : \kappa e : 1$$
$$:: \tfrac{1}{2}p^2-2q+4s^{\frac{1}{2}} : 4\,(ps^{\frac{1}{2}}-r) : 2qs-4s^{\frac{3}{2}}-\tfrac{1}{2}r^2 : 4\,(ps-rs^{\frac{1}{2}}) : 4qs^{\frac{1}{2}}-8s-rp.$$

The given bicircular may take these values of its constants:

$$\lambda\,(x^2+y^2)^2+(1+fx)\,(x^2+y^2)=\kappa\,(1+ex).$$

The distances of the apses from the double focus at the origin may be expressed implicitly in terms of the distances of the four single foci:

$$(p^2-4q+8s^{\frac{1}{2}})\,x^4+8\,(ps^{\frac{1}{2}}-r)\,x^3+2\,(4qs^{\frac{1}{2}}-8s-rp)\,x^2$$
$$=8\,(ps-rs^{\frac{1}{2}})\,x+4qs-8s^{\frac{3}{2}}-r^2.$$

This quartic has the same discriminant as $1+p\xi+q\xi^2+r\xi^3+s\xi^4=0$, since two foci unite at a node.

The roots of this quartic are given in § 11,

$$x\,(\sqrt{a}\pm\sqrt{b}\pm\sqrt{c}\pm\sqrt{d})+\sqrt{(abcd)}\,\left(\frac{1}{\sqrt{a}}\pm\frac{1}{\sqrt{b}}\pm\frac{1}{\sqrt{c}}\pm\frac{1}{\sqrt{d}}\right)=0.$$

15. For the circular cubic, $p^2-4q+8s^{\frac{1}{2}}=0$. Its equation is found after rejecting $ps^{\frac{1}{2}}-r$,

$$(8x+2p)\,(x^2+y^2)=8s^{\frac{1}{2}}x+ps^{\frac{1}{2}}+r,$$

where the focal distances have the relation

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}=0.$$

If $d = 0$, or the quartic have a triple focus at the origin, the quartic of § 14 becomes $(p^2-4q)\,x^4-8rx^3-2rpx^2+r^2 = 0$,

whose roots are $x\,(\pm\sqrt{a}\pm\sqrt{b}\pm\sqrt{c}) = \sqrt{abc}$.

Other modifications occur when other terms in the quartic are removed.

I next proceed to obtain relations between two parameters in the quartic, which express general properties in the form of curves.

16. To obtain the relation between the parameters, when bicircular quartics with collinear foci are resolved into pairs of circles.

Resume the ordinary form of these quartics (§ 13),

$$\lambda r^4 + r^2\,(1+bx) = \kappa\,(1+ax).$$

This will be a rational function of r^2 or x^2+y^2, if

$$(1+bx)^2 + 4\kappa\lambda\,(1+ax)$$

be a complete square.

The required relation is expressed by the hyperbolæ,

$$\kappa\lambda\,(\kappa\lambda a^2 - b^2 + ab) = 0.$$

17. To determine the first discriminating curve, or the mutual relation between the parameters, when the bicircular quartics in this group are singular. (Fig. 1.)

At a singular point $\dfrac{d\phi}{dx} = 0, \quad \dfrac{d\phi}{dy} = 0.$

For the quartic $\lambda r^4 + r^2\,(1+bx) - \kappa\,(1+ax) = 0 \equiv \phi,$

$$\kappa a = 2x\,(1+bx) + 4\lambda x r^2 + br^2,$$
$$0 = 2y\,(1+bx) + 4\lambda y r^2.$$

Hence also $\kappa\,(4+3ax) = (2+bx)\,r^2.$

[The second condition is resolved into two parts

$$1+bx+2\lambda r^2 = 0 \ldots\ldots (1); \text{ and } y = 0 \ldots\ldots (2).$$

The first yields an hyperbola as a portion of curve (D_1),

$$\kappa\lambda a^2 + b\,(a-b) = 0,$$

and all nodes lie in the line $abx + 2b + a = 0$.

The second part gives a sextic as a second portion.

$$4\lambda x^3 + 3bx^2 + 2x - \kappa a = 0,$$
$$bx^3 + 2x^2 - 3\kappa a x - 4\kappa = 0.]$$

The parameters κ, λ are connected, as the coordinates of a unicursal sextic, or quintic, if $a = 0$. The equation may be obtained, without the intermediary (x), as the discriminant of the binary quartic (ϕ), when $y = 0$.

Besides the axes of coordinates, there is a linear asymptote

$$27a^4\kappa + 256\lambda = 48a\,(4b-3a),$$

which corresponds to the value $\left(-\dfrac{4}{3a}\right)$ of x; but not when $a = \infty$.

At singular points on the sextic

$$\frac{d\kappa}{dx} = 0, \quad \frac{d\lambda}{dy} = 0.$$

These conditions determine two values of x,

$$3abx^2 + 6bx + 3ax + 8 = 0,$$

to each of which a cuspidal value of (κ, λ), and thereby a corresponding cusped quartic, corresponds, if the values of (x) are real. Various modifications occur, when a and b have the values zero or infinity.

18. The cases should be noticed when these values of x unite. When this occurs, (1) $a = 6b$, and (2) $3a = 2b$.

For (1) the cusps unite in a point of undulation, and for (2) in a stationary point.

The corresponding quartics have, the one a cusp, and the other a folium-point, but they are reduced to points.

For the value (2), the sextic curve (D_1) degenerates into the quartic

$$(12\kappa\lambda + 3)^2 = 32b^2\kappa \ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

and the line

$$48\lambda + b^4\kappa = 12b^2 \ldots\ldots\ldots\ldots\ldots\ldots \text{(B).*}$$

19. To determine the second discriminating curve, or the mutual relation between the parameters, when bicircular quartics with collinear foci have points of undulation.

This curve (D_2) consists of two portions, obtained by different processes. The ordinary tests for undulation will be first taken.

I. At a folium-point, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} = 0$.

Let these tests be applied to the quartic,

$$\lambda r^4 + (1 + bx)\, r^2 = \kappa\,(1 + ax),$$

or, when solved,

$$2\lambda r^2 + (1 + bx) = \sqrt{\{(1 + bx)^2 + 4\kappa\lambda\,(1 + ax)\}} \equiv 2\sqrt{U}.$$

By successive differentiation,

$$2\lambda \frac{d}{dx}(r^2) + b = U^{-\frac{1}{2}} \frac{dU}{dx},$$

$$2\lambda \frac{d^2}{dx^2}(r^2) = U^{-\frac{1}{2}} \frac{d^2U}{dx^2} - \tfrac{1}{2} U^{-\frac{3}{2}} \left(\frac{dU}{dx}\right)^2,$$

$$0 = -\tfrac{3}{2} U^{-\frac{3}{2}} \frac{dU}{dx} \frac{d^2U}{dx^2} + \tfrac{3}{4} U^{-\frac{5}{2}} \left(\frac{dU}{dx}\right)^3.$$

This condition is resolved into two factors,

$$2U \frac{d^2U}{dx^2} - \left(\frac{dU}{dx}\right)^2 = 0, \qquad \frac{dU}{dx} = 0.$$

---

* The quartics, whose parameters are thus related (B), are singular, but not cuspidal.

The first factor gives the relation of § 16, when the quartic is resolved into two circles. We are concerned with the second factor,

$$b\,(1+bx)+2\kappa\lambda a = 0.$$

In this case,

$$r^2 = \frac{\kappa a}{b} \pm \frac{1}{\lambda}\,U^{\frac{1}{2}} = \frac{\kappa a}{b} \pm \sqrt{\left\{\frac{\kappa}{\lambda}\left(1-\frac{a}{b}\right)-\frac{\kappa^2 a^2}{b^2}\right\}},$$

$$2\lambda\,\frac{d}{dx}(r^2)+b=0, \quad \text{or} \quad x+y\frac{dy}{dx}+\frac{b}{4\lambda}=0,$$

$$2\lambda\,\frac{d^2}{dx^2}(r^2) = \frac{b^2}{2}\,U^{-\frac{1}{2}}, \quad \text{or} \quad 1+\left(\frac{dy}{dx}\right)^2 = \frac{b^2}{8\lambda}\,U^{-\frac{1}{2}}.$$

The last may take the form

$$y^2+\left(x+\frac{b}{4\lambda}\right)^2 = \frac{b^2y^2}{8\lambda}\,U^{-\frac{1}{2}},$$

or

$$r^2+\frac{bx}{2\lambda}+\frac{b^2}{16\lambda^2} = \frac{b^2}{8\lambda}\,(r^2-x^2)\,U^{-\frac{1}{2}}.$$

That is,

$$\frac{1}{\lambda}\,U^{\frac{1}{2}}-\frac{1}{2\lambda}+\frac{b^2}{16\lambda^2} = \left\{\frac{\kappa a}{b}+\frac{1}{\lambda}\,U^{\frac{1}{2}}-\frac{1}{b^4}\,(2\kappa\lambda a+b)^2\right\}\frac{b^2}{8\lambda}\,U^{-\frac{1}{2}},$$

or

$$8U-4U^{\frac{1}{2}}-\frac{b^2}{2\lambda}\,U^{\frac{1}{2}} = \kappa ab-\frac{1}{b^2}\,(2\kappa\lambda a+b)^2.$$

After further reduction, this part of curve (D_2) is thus defined :

$$\left\{1-\kappa ab+4\left(2-\frac{a}{b}\right)\kappa\lambda-\frac{4a^2}{b^2}\,\kappa^2\lambda^2\right\}^2$$

$$= \frac{1}{b^2}\,\kappa\lambda\,(b^2-ab-a^2\kappa\lambda)\left(4+\frac{b^2}{2\lambda}\right)^2. \quad \text{(Fig. 2.)}$$

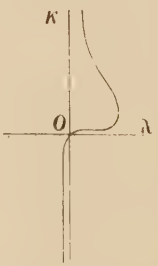


Fig. 2.

The curve has only one branch, with three inflexions, and (since it depends on the product κ, λ) lies in the first and third, or in the second and fourth quadrants, according as $a <> b$. The curve has two acnodes, whose abscissa is $-\frac{b^2}{8\lambda}$.

If $a = 0$ or b, this portion of curve (D_2) does not exist.

II. The preceding process is applicable to folium-points in companion-curves to lemniscatoids (see § 20) ; but for the neighbourhood of limaçonoids the conditions may be thus found.

The bitangents touch the apses in such folium-points.

The conditions are

$$4U = (1+bx)^2+4\kappa\lambda\,(1+ax) = 0,$$

and $\quad 2\lambda x^2+1+bx = 0, \qquad$ since $y = 0$.

The eliminant is a quintic. (Fig. 3.)

$$\lambda\,(1-4\kappa\lambda)^2$$

$$= -\,2\kappa\lambda b\,(3a-4b)+\kappa b^3(a-b)-8\kappa^2\lambda^2 a\,(a-b).$$

If $a = 0$, the figure has no asymptote but the axes.*

$a = b$, curve (D_2) becomes a bicusped quartic, as in Fig. 4.

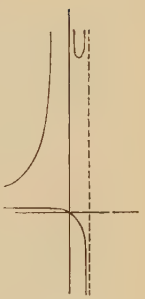


Fig. 3.

20. To determine all bicircular quartics with collinear foci by the aid of the two discriminating curves.

The singular quartics are crunodal and acnodal. The former are lemniscatoid for values of (κ, λ) in the first quadrant, and limaçonoid in the second quadrant. The acnodal quartics shrink to acnodes for points (κ, λ) on the bounding lines NQ, PL; for ulterior values of (κ, λ) no corresponding quartics are possible. (Fig. 1.)

The quartics may be bi-, uni-, and non-folium.

If (κ, λ) lie between the curve (D_2) of Fig. 2, the quartic is bifolium; if (κ, λ) lie between the branches of curve (D_2) of Fig. 3, the quartic is unifolium. For values of (κ, λ) exterior to the branches of Figs. 2 and 3, the corresponding quartics are non-folium.

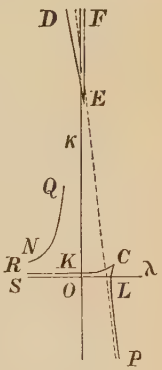


Fig. 1.

The sequence has been minutely exhibited in *Proceedings*, Vol. xii., pp. 21, 22, with illustrations, for the several quadrants, with the omission of the limitations created by the second branch (Fig. 3) of curve (D_2).

The cuspidal portion *DEF* of Fig. 1 protrudes through the apsidal portion of Fig. 3; the corresponding quartics have the same character as for points in and near the cuspidal part *KCL*.

It is easy to anticipate the number of folia in the neighbourhood of critical quartics, by considering the shapes of these last. Near lemniscatoids, the companion-curves are bifolium; near limaçonoids, unifolium.

21. For two relative values of the constants, the first discriminating curve is modified, as was pointed out in § 18.

* In the description of the curve (D_2) (*Proceedings*, Vol. xii., p. 25), this portion was omitted. Without its aid, the limiting forms of Fig. 4 (p. 22) cannot be found.

If $3a = 2b$, $2\kappa = x^2$, $6\lambda x^2 + 4bx + 3 = 0$.

The eliminant defines curve (D_1) as the bicusped quartic $(12\kappa\lambda + 3)^2 = 32\kappa b^2$. (Fig. 4.)

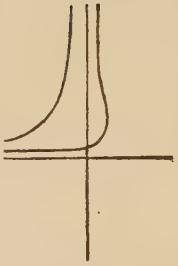


Fig. 4.

In the second quadrant, the sequence of values of the corresponding quartics is unaltered; the two cusps of Fig. 1 have united in a point of inflexion. In the first quadrant, critical quartics are acnodal, and the companion-curves are bifolium, according to the position of (κ, λ) on curve (D_2). (Fig. 2.)

If $a = 6b$, the sextic is also modified: the two cusps of Fig. 1 have united in a point of undulation. (Fig. 5.)

Fig. 5.

The quartics, which correspond to the point of inflexion in Fig. 4, and the folium-point in Fig. 5, are cuspidal and undulational, but both mere point-quartics, as these points (κ, λ) are on the limiting branches of curve (D_1),

$$5b^4y^4 + 2b^2y^2(5b^2x^2 + 12bx + 12) + (bx+2)^3(5bx-6) = 0,$$

$$81b^4y^4 + 54b^2y^2(3b^2x^2 + 4bx + 4) + (3bx+2)^4 = 0.$$

22. In the case, when the bicirculars pass through the origin,

$$\lambda(x^2+y^2)^2 + (1+bx)(x^2+y^2) = \kappa x.$$

The curve (D_1) is the tricuspidal quartic (see § 17)

$$(b+9\kappa\lambda)^2 = 4(b^2-3\lambda)(1+3b\kappa).$$

[To values of (κ, λ) on curve (D_1) in the first quadrant there correspond acnodal quartics: in the fourth quadrant, as (κ, λ) on curve (D_1) approaches the cusp, an acnodal quartic and a lemniscatoid gradually unite in the cusped quartic

$$b^3y^2(y^2+2x^2) + 3by^2(1+bx) + x(bx+1)^3 = 0.]$$

To critical values in the second and third quadrants, limaçonoids correspond.

23. In the case where both $\frac{1}{a}$ and $\frac{1}{b} = 0$,

or $$\lambda(x^2+y^2)^2 + bx(x^2+y^2) = \kappa x,$$

the curve (D_1) is the cubic hyperbola $27\kappa\lambda^2 = 4b^3$,

the curve (D_2) is the homothetic cubic $8\kappa\lambda^2 = b^3$,

since the other portion (§ 19, II.) does not exist.

If the coordinates be considered tangential, this group comprises those class-quartics which have a triple focus and a single focus at infinity, and whose satellite-conics have the same as single foci. The critical quartics are acnodal; the companion-curves, whether single,

or pairs of ovals, non-folium or uni-folium as (κ, λ), are above or below curve (D_2). The origin is in all cases a folium-point.

[If $\frac{1}{b} = 0$ only, there is a cusp on curve (D_1), and the corresponding quartic is resolved into two circles

$$a^2(x^2+y^2) = 4, \quad (ax+2)^2+y^2 = 0.$$

The case when $a = 0$ has been studied (Vol. xii., pp. 17–27); Cartesians, in which $b = 0$, will be examined in § 29.]

24. Bicircular quartics, which have the same single foci, but distinct double foci, are mutually orthotomic.

This is an extension of Prof. Crofton's theorem for Cartesians (*Proceedings*, Vol. i., 6).

Let the vector equation to one of the group be

$$lr_1+mr_2+nr_3 = 0,$$

where (by § 10) $l : m : n :: (b-c)\{\sqrt{(ad)}-\sqrt{(bc)}\}$
$: (c-a)\{\sqrt{(bd)}-\sqrt{(ca)}\}$
$: (a-b)\{\sqrt{(cd)}-\sqrt{(ab)}\}.$

The equation to a tangent circle at the point (R_1, R_2, R_3) is

$$\frac{lr_1^2}{R_1}+\frac{mr_2^2}{R_2}+\frac{nr_3^2}{R_3} = 0.$$

Then another quartic of the group may be written

$$l'r_1+m'r_2+n'r_3 = 0,$$

where $l' \propto (b-c)\{\sqrt{[(a+h)(d+h)]}-\sqrt{[(b+h)(c+h)]}\}$,

the triple focus being moved, but r_1, r_2, r_3 being unaltered.

Its tangent circle is $\frac{l'r_1^2}{R_1}+\frac{m'r_2^2}{R_2}+\frac{n'r_3^2}{R_3} = 0.$

The condition that the tangent circles, or, as a consequence, that the two quartics intersect orthogonally, is

$$2\left(\frac{la}{r_1}+\frac{mb}{r_2}+\frac{nc}{r_3}\right)\left(\frac{l'a}{r_1}+\frac{m'b}{r_2}+\frac{n'c}{r_3}\right)$$
$$= \left(\frac{l}{r_1}+\frac{m}{r_2}+\frac{n}{r_3}\right)\left(\frac{l'a^2}{r_1}+\frac{m'b^2}{r_2}+\frac{n'c^2}{r_3}\right)$$
$$+\left(\frac{l'}{r_1}+\frac{m'}{r_2}+\frac{n'}{r_3}\right)\left(\frac{la^2}{r_1}+\frac{mb^2}{r_2}+\frac{nc^2}{r_3}\right).$$

This condition may be simplified:

$$\frac{1}{r_2 r_3}(mn'+m'n)(b-c)^2+\frac{1}{r_3 r_1}(nl'+n'l)(c-a)^2 + \frac{1}{r_1 r_2}(lm'+l'm)(a-b)^2=0.$$

But, from the two quartics, r_1, r_2, r_3 are the minors of the determinant

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \end{vmatrix}.$$

The condition of orthotomy becomes

$$\Sigma(m^2n'^2-m'^2n^2)(b-c)^2=0.$$

It may be proved that this is an identity.

Substitute for m, n, m', n', their values, and reject the common factor $(b-c)^2(c-a)^2(a-b)^2$,

$$\{\sqrt{(bd)}-\sqrt{(ca)}\}^2\{\sqrt{[(c+h)(d+h)]}-\sqrt{[(a+h)(b+h)]}\}^2,$$
$$-\{\sqrt{(cd)}-\sqrt{(ab)}\}^2\{\sqrt{[(b+h)(d+h)]}-\sqrt{[(c+h)(a+h)]}\}^2,$$

together with two other sets of values, may be shown to be zero.

The collected values of the sets may be reduced to the factors

$$\{(bd+ac-cd-ab)+(cd+ab-ad-bc)+(ad+bc-bd-ac)\},$$
$$\{h(a+b+c+d)+2h^2+2\sqrt{(abcd)}-2\sqrt{[(a+h)(b+h)(c+h)(d+h)]}\};$$

of which one is identically zero.

The general problem is considered in *Higher Plane Curves*, p. 238.

25. To generate a Cartesian by Dr. Casey's method of reciprocation.

It will be noticed that the method of § 4 fails for generating Cartesians, since $d=0$, when a double becomes a triple focus.

In this case, three concentric circles take the place of confocal conics:

$$x^2+y^2=A^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (F_1),$$
$$x^2+y^2=B^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (F_2),$$
$$x^2+y^2=C^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (F_3);$$

and there are three corresponding circles of reciprocation, which are mutually orthotomic:

$$(x-f)^2+y^2=\rho_1^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (J_1),$$
$$(x-g)^2+y^2=\rho_2^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (J_2),$$
$$(x-h)^2+y^2=\rho_3^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (J_3).$$

The Cartesian is generated in three ways as the envelope of the variable circle $(x-\alpha)^2+(y-\beta)^2=\gamma^2$,

which is subject to the conditions that its centre is in F,

$$\alpha^2+\beta^2=A^2,$$

and that it cuts J orthogonally,

$$2af-a^2-\beta^2+\gamma^2+\rho_1^2-f^2=0.$$

The Cartesian envelope is found to be

$$(x^2+y^2)^2+2(\rho_1^2-f^2-2A^2)(x^2+y^2)+8A^2fx+(\rho_1^2-f^2)^2-4A^2f^2=0.$$

Dr. Casey has been anticipated in this method of generating Cartesians by Quetelet (Chasles, *Aperçu Historique*, p. 350), if in his theorem we measure the distance from the centre of the variable circle to the circumference of the orthotomic circle along a tangent.

In like manner, its equation is found in terms of the constants

$$\rho_2,\ g,\ B\ ;\quad \rho_3,\ h,\ C.$$

By equating the coefficients of like terms in these three forms of the same Cartesian,

$$\rho_1^2=f^2-fg-B^2+A^2=f^2-fh-C^2+A^2,$$

$$A^2f=B^2g=C^2h=fgh.$$

The (F) circles are

$$x^2+y^2=gh,\quad x^2+y^2=fh,\quad x^2+y^2=fg.$$

The (J) circles are $(x-f)^2+y^2=(f-g)(f-h)$,

$$(x-g)^2+y^2=(g-f)(g-h),$$

$$(x-h)^2+y^2=(h-f)(h-g).$$

The Cartesian thus generated is defined by the form

$$(x^2+y^2)^2-2(gh+hf+fg)(x^2+y^2)$$
$$+8fghx+h^2g^2+f^2h^2+f^2g^2-4fgh(f+g+h)=0.$$

It will appear (§ 28) that f, g, h are the distances of the single foci from the triple focus at the origin.

26. The dual theorem may be noticed.

If the variables be interpreted as Boothian tangential coordinates, the (F) equations still denote circles, and the (J) equations singly confocal conics, with parallel directrices.

Thus the class-quartic

$$\kappa(1+e\xi)=\xi^2+\eta^2+\lambda(\xi^2+\eta^2)^2,$$

which has a triple and a single focus, and for which the centre of its satellite-circle coincides with the triple focus, may be generated in three ways.

It is the envelope of a variable conic, with a fixed focus, whose directrix touches a fixed dirigent circle (F), and which has a common tangent with a fixed confocal conic (J), such that the points of contact subtend a right angle at the common focus.

27. The distances of the apses from the triple focus are found by writing $y = 0$, in the Cartesian of § 25.

The reducing cubic of this binary quartic by Euler's method is

$$z^3 - qz^2 + prz - r^2 = 0.$$

Its roots are gh, fh, fg, the squares of the radii of the (F) circles. (Casey, p. 491.)

Both quartic and cubic have the same discriminant

$$(g-h)^2 (h-f)^2 (f-g)^2.$$

28. The equation to the bicircular (§ 25) may assume various forms

$$(x^2+y^2+gh-hf-fg)^2 = 4gh\{(x-f)^2+y^2\}.$$

Hence, if ρ_1, ρ_2, ρ_3 denote the focal distances of a current point,

$$2\rho_1\sqrt{(gh)} = x^2+y^2+gh-hf-fg;$$

so also

$$2\rho_2\sqrt{(hf)} = x^2+y^2+hf-fg-gh,$$

$$2\rho_3\sqrt{(fg)} = x^2+y^2+fg-gh-hf.$$

Again, by subtracting these equations in pairs, we obtain the usual vector forms

$$h\sqrt{g}\rho_2 - g\sqrt{h}\rho_3 = \sqrt{(fgh)}\,(h-g),$$

$$f\sqrt{h}\rho_3 - h\sqrt{f}\rho_1 = \sqrt{(fgh)}\,(f-h),$$

$$g\sqrt{f}\rho_1 - f\sqrt{g}\rho_2 = \sqrt{(fgh)}\,(g-f).$$

Lastly, by addition, the equation of § 10, Cor. I., is derived:

$$(g-h)\sqrt{f}\rho_1 + (h-f)\sqrt{g}\rho_2 + (f-g)\sqrt{h}\rho_3 = 0.$$

(Casey, p. 486.)

29. To determine the foci in Cartesians.

The equation to the group is taken to be

$$\lambda(x^2+y^2)^2 + (x^2+y^2) = \kappa(1+ex).$$

Since the circular points at infinity are cusps, Cartesians are class-sextics, and not class-octavics,* as ordinary bicircular quartics are. The origin is a triple focus, and there are three single foci.

Proceed to find the equivalent class-sextic as in § 13, when $f = 0$,

$$S^3 - 27T^2 = 0.$$

If the terms be omitted at first from S and T, which contain κ, the expanded form is measured by $(\xi^2+\eta^2)^3$, or the class is diminished from 12 to 6 dimensions.

The residue, after rejecting this factor R^3, is the equivalent class-

* In the second edition of the *Aperçu Historique*, p. 353 (Paris, 1875), the error is repeated, which was long ago corrected by Prof. Cayley, *Liouville*, Vol. xv., p. 354.

sextic $16\kappa^2\lambda^3e^2+8\kappa\lambda^2e\,(1+4\kappa\lambda)\,\xi+16\kappa^2\lambda^2e^2\xi^2\,(1-\kappa e\xi)$
$+(\xi^2+\eta^2)\,\{\lambda\,(1+4\kappa\lambda)^2+8\kappa\lambda\,(1+4\kappa\lambda)\,e\xi\,(1-\kappa e\xi)-4\kappa^3\lambda^2e^2\xi^2\}$
$+(\xi^2+\eta^2)^2\,\{(1+4\kappa\lambda)^2\,(1-\kappa e\xi)-8\kappa^2\lambda\,(1+4\kappa\lambda)\,e\xi\}$
$-(\xi^2+\eta^2)^3\,\kappa\,(1+4\kappa\lambda)^2$
$=-6\kappa^2\lambda^2e^2\eta^2\,(4-3\kappa e\xi)$
$+(\xi^2+\eta^2)\,\{\tfrac{2}{1}\tfrac{7}{6}\kappa^4\lambda e^2\eta^4-\kappa^2\lambda e^2\eta^2\,(8-9\kappa e\xi)+\tfrac{1}{2}\kappa^2\lambda e^2\eta^2\,(1+36\kappa\lambda)\}$
$+(\xi^2+\eta^2)^2\,(1+36\kappa\lambda)\,\tfrac{1}{4}\kappa^2e^2\eta^2.$

The focal lengths are obtained by substituting $(\xi^2+\eta^2)-\xi^2$ for η^2, and rejecting all the terms which involve $\xi^2+\eta^2$,

$$8\kappa\lambda e+4\,(1+4\kappa\lambda)\,\xi-4\kappa e\xi^2+\kappa^2e^2\xi^3=0.$$

Its discriminant, $27\kappa^3\lambda e^4+4\kappa e^2\,(1+36\kappa\lambda)+16\,(1+4\kappa\lambda)^3$,
is the same as that of the expression for the apsidal distances

$$\lambda x^4+x^2-\kappa ex-\kappa=0.$$

30. To determine the first discriminating curve, when Cartesians have singular points. (Fig. 6.)

This curve (D_1) is defined by the discriminant of the binary quartic of § 29 $(y=0)$ as a bicusped quartic

$$16\,(1+4\kappa\lambda)^2+4\kappa e^2\,(1+36\kappa\lambda)+27\kappa^3\lambda e^4=0.$$

It is drawn more simply as a unicursal curve by the conditions of § 17, by equating b to zero,

$$\kappa=\frac{2x^2}{4+3ex},\quad -\lambda=\frac{ex+2}{x^2\,(4+3ex)}.$$

The linear asymptote has the form

$$27e^4\kappa+256\lambda+144e^2=0.$$

This quartic in (κ, λ) has two cusps, one at infinity, and the other $\left(-\frac{32}{9e^2},\ -\frac{3e^2}{128}\right)$, when $3ex+8=0$.

Fig. 6.

To these related values of κ, λ, limaçon Cartesians correspond; and to the finite cusp the cardioid,

$$27e^4y^4+18e^2y^2\,(3e^2x^2-64)+(ex-8)\,(3ex+8)^3=0.$$

There are no lemniscatoid Cartesians, since no part of curve (D_1) is in the first quadrant.

31. To determine the second discriminating curve, when Cartesians have points of undulation. (Fig. 7.)

The second process of § 19 is alone applicable.

The conditions, that the bitangents touch the apses in folium-points, are

$$1+4\kappa\lambda\,(1+ex)=0,\quad 2\lambda x^2+1=0.$$

The eliminant defines curve (D_2) as the quartic $(1+4\kappa\lambda)^2+8\kappa^2\lambda e^2=0$, and the (κ) axis.

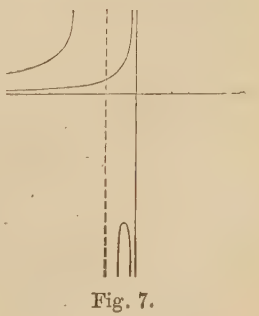


Fig. 7.

The abscissa is essentially negative, and the ordinate has a minimum value. For values of (κ, λ) within the branches of Fig. 7 the corresponding quartics have veri-bitangents; for exterior values of (κ, λ), acu-bitangents.

32. By the aid of these two discriminating curves, all Cartesians may be exhibited.

In the first quadrant of Figs. 6, 7, there is no singular Cartesian, nor has any curve a folium-point. Each is an oval without a folium, and has an acu-bitangent.

In the second quadrant, for values of (κ, λ) on or above the upper branch of Fig. 6, the corresponding Cartesians are points or do not exist; for values below and above the lower branch of Fig. 6, the Cartesians are single ovals. If (κ, λ) is on this branch of Fig. 6, the curves are limaçons, and for lower values, two ovals.

The single ovals have imaginary or real bitangents, as (κ, λ) is above or below the branch of Fig. 7; the pairs of ovals have veri- or acu-bitangents, as (κ, λ) is above or below the lower branch, terminating in two circles, when $\kappa = 0$.

In the third quadrant, the Cartesians consist of an oval with an acu-bitangent, unless (κ, λ) falls within curve (D_2).

If (κ, λ) is within the loop, and beyond the cusp of Fig. 6, the Cartesian (which protrudes in the former loop) is a single veri-bitangential oval.

If (κ, λ) is within the cuspidal part of Fig. 6, the Cartesian has a pair of ovals, terminating for nodal values in a limaçon and an acnodal quartic. If (κ, λ) be the cusp, the nodal values unite in the cusped cardioid.

In the fourth quadrant, no Cartesian is possible for values of (κ, λ) above curve (D_1). For values of (κ, λ) within that curve and the (κ) axis, the curves have a single acu-bitangential oval, commencing with acnodes and ending in pairs of circles.

33. If the origin is on the Cartesian, its equation becomes

$$\lambda (x^2+y^2)^2+x^2+y^2 = \kappa x.$$

The curves (D_1), (D_2) are two homothetic cubics:

$$27\kappa^2\lambda+4 = 0, \quad 8\kappa^2\lambda+1 = 0.$$

Cartesians occur for all values of the parameters. For points (κ, λ) beyond both branches of curve D_1, they are single ovals with veri-bitangents; for intermediate values of (κ, λ), the Cartesians consist of a pair of ovals, with real or imaginary bitangents, as (κ, λ) is beyond or within curve (D_2).

34. If $e = 0$, the Cartesian is resolved into two circles.

35. The classification of this memoir is applicable to those cyclides, which are generated by the revolution about their axes of bicircular quartics with collinear foci.

On Cremonian Congruences. By Dr. T. Archer Hirst, F.R.S.

[*Read May 13th*, 1880.]*

Definition.

1. The well-known point-to-point correspondence between two planes, which was first investigated by Cremona in two Memoirs, published in the Transactions of the Institute of Bologna for 1863 and 1865,† is now usually referred to as a Cremonian Correspondence. The two-fold system of right-line rays, each of which passes through a pair of corresponding points of two planes thus related, but arbitrarily situated in space, may therefore be appropriately termed a *Cremonian Congruence.*

2. Although far from being always general ones of their order and class, Cremonian Congruences, from their very mode of generation, cannot fail to secure the interest of Geometers. This interest is enhanced, moreover, by the fact that they include many well-known and important types, whose direct investigation, by purely geometrical methods, has hitherto received comparatively little attention.

The present paper is occupied mainly with the study of Cremonian Congruences, whose class, or whose order, does not exceed the second; the general properties of Cremonian Congruences will be treated in it only so far as this main object of the paper may appear to demand.

General Properties.

3. Let α and β be the two planes between which a Cremonian Correspondence of the n^{th} degree is supposed to exist. An arbitrary plane π will cut each of them in a line to which a curve of the order n corresponds, point by point, in the other. Every such plane π, therefore, will in general contain n pairs of corresponding points, and consequently n rays of the Cremonian Congruence;—in other words, *this congruence is of the class n.*

4. Should the plane π pass through the intersection of the two generating planes α and β, without coinciding with either of them, all the congruence-rays in it will, in general, be coincident with $\alpha\beta$; hence *the intersection of the planes α and β is a n-ple ray of the Cremonian Congruence.*

---

* The publication of this paper has been unavoidably delayed for more than three years; the form in which it now appears is, substantially, that which it had when communicated, in abstract, to the Society.

† Both memoirs were afterwards republished in the *Bulletin des Sciences Mathématiques et Astronomiques* (t. v., p. 206, 1873), to which, under the designation *Bulletin*, reference will be hereafter made.

5. When the plane π, through $\overline{\alpha\beta}$, coincides with either of the generating planes, it obviously contains the singly infinite number of congruence-rays connecting the several points of $\overline{\alpha\beta}$ with the points in that plane which respectively correspond to them. *The generating planes α and β, therefore, are singular planes of the congruence under consideration.*

6. The several pairs of corresponding points of α and β are projected from an arbitrary point P in space by two pencils between which a Cremonian correspondence of the n^{th} degree likewise exists. The $n+2$ self-corresponding rays which these concentric pencils possess* are evidently congruence-rays, and, moreover, the only ones passing through P; in other words, *the Cremonian Congruence generated by the planes α and β is of the order $n+2$.*

7. Of the $n+2$ congruence-rays which pass through an arbitrary point of α, or β, however, all but one necessarily lie in that plane. The excepted one connects the point in question with its correspondent in β, or α; the remaining $n+1$ connect points of β, or α, situated on $\alpha\beta$, with their correspondents in α or β. Hence *the congruence-rays in each of the two singular planes α and β envelope a congruence-curve of the class $n+1$, to which*, by Art. 4, $\overline{\alpha\beta}$ *is a n-ple tangent.* These congruence-curves will be denoted by a_{n+1} and b_{n+1} respectively; being unicursal, they are, in general, of the order $2n$.

8. The correspondents, above alluded to, of the several points of $\overline{\alpha\beta}$ lie, by hypothesis, on a curve $\overline{a}^n$ in α, and $\overline{b}^n$ in β. These curves, being unicursal and of the order n, are in general of the class $2(n-1)$. They cut $\overline{\alpha\beta}$ in the n pairs of corresponding points which this line contains; each, in fact, cuts it in the n points at which it is touched by the congruence-curve situated in the plane of the other, as will be evident on remembering that every tangent of the congruence-curve in α, or β, joins a point on $\overline{\alpha\beta}$ to its correspondent on $\overline{a}^n$, or $\overline{b}^n$, (Art. 7). The curves $\overline{a}^n$ and a_{n+1}, moreover, (and in like manner $\overline{b}^n$ and b_{n+1}) have contact with each other at every point where the former is touched by the tangent of the latter which connects that point with its correspondent on $\overline{\alpha\beta}$. It may be readily shown, by Chasles' *Principle of Correspondence*, that the number of such points of contact is in general $2n-1$.

9. The fixed (principal) points through which every curve, of the order n, corresponding to a right line passes, once or oftener, are *singular points* of the Cremonian Congruence. Each, in fact, is the

* *Bulletin*, Art. 35.

correspondent, not of one point solely, but of a singly infinite number of points, all of which are situate on a (principal) line or unicursal curve whose order is equal to the multiplicity of the principal point in question, and whose own multiple points are themselves coincident with principal points.* Through every principal point of either of the two planes α or β, therefore, a singly infinite number of congruence-rays pass, all of which are situated on a *congruence-cone*, whose trace on β or α is the principal curve corresponding to its vertex.

The multiple generators of each such cone are obviously congruence-rays of the same order of multiplicity. If A_i, for instance, be a principal point of α, of the order i, whose corresponding principal curve b^i passes k times through a principal point B_j of β, then, as Cremona has shown,† the principal curve a^j, corresponding to the latter point, will pass k times through A_i. Hence it follows that, of the n congruence-rays situated in any plane passing through both these points, k, at least, must coincide with $\overline{A_iB_j}$ itself.

The singularity would be still more marked if either of the points A_i or B_j, or both, were situated on the intersection $\overline{\alpha\beta}$; but I do not pause to consider it.

10. The class of the focal surface of a Cremonian Congruence is readily determined from the circumstance that every tangent plane thereof (since it contains, by definition, two infinitely close congruence-rays) must, in general, intersect α and β in lines which touch their corresponding curves.

Now, a plane π, by turning around an arbitrary line p, generates two pencils of rays $[p\alpha]$ and $[p\beta]$, in α and β, which are in perspective with each other, as well as projective with the pencils of curves, of order n, corresponding to their several rays. The two projective pencils in each plane, say in α, generate, by the intersection of their associated elements, a curve a^{n+1} of the order $n+1$, which passes once through $(p\alpha)$ and the correspondent of $(p\beta)$, and i times through every principal point A_i. The class of the generated curve, therefore, is in general

$$n(n+1)-\Sigma i(i-1)a_i = 4n-2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where a_i denotes the number of principal points in α of the order i, and it is to be remembered that

$$\Sigma i^2 a_i = n^2-1, \quad \text{and} \quad \Sigma i a_i = 3(n-1),$$

if the summation be extended from $i=1$ to $i=n-1$.‡ By a well-known theorem, the tangent to the curve a^{n+1} at the point $(p\alpha)$ is the ray of the pencil $[p\alpha]$ associated with the curve, of the projective

---

* *Bulletin*, Art. 5. † *Bulletin*, Art. 12. ‡ *Bulletin*, Arts. 4 and 8.

pencil, which passes through $(p\alpha)$*; but this ray and curve do not, in general, touch each other at $(p\alpha)$; whereas every ray of the pencil $[p\alpha]$ which touches a^{n+1} at a point not coincident with $(p\alpha)$, likewise touches its associated curve at that point. Hence it follows that the plane through p and any such ray possesses the property, above referred to, of intersecting α and β in lines which touch their corresponding curves, and therefore of touching the focal surface. According to the equation (1), the number of such planes passing through p, in other words *the class of the focal surface, is* $4(n-1)$.

11. If p were an ordinary congruence-ray,—that is to say, if it intersected α and β in a pair of corresponding points, neither being a principal point,—then the curves associated with the several rays of the pencil $[p\alpha]$ would pass through its centre, and the generated curve a^{n+1} would there acquire an additional double point, the tangents at which would touch the curves respectively associated with them.† Although the class of a^{n+1} would be diminished by 2, therefore, every plane of the pencil $[p]$ by which it is touched would become a tangent plane of the focal surface, and moreover, with each of the planes touching it at its double point, two consecutive tangent planes of that surface would coincide; in short, as is well known, every congruence-ray p is a double tangent of the focal surface; the two planes in question are its planes of contact—*focal planes*, according to Kummer.‡

12. The only tangent planes to the focal surface which pass through an arbitrary line p situated in either of the generating planes α and β, and which do not coincide with that plane, are those which touch the curve corresponding to p. Their number is, in general, $2(n-1)$ (Art. 8); whence, and from the fact that the focal surface is of the class $4(n-1)$ (Art. 10), we infer that $2(n-1)$ *of the tangent planes to the focal surface which pass through an arbitrary line of either of the two generating planes coincide with that plane.*

13. As multiple tangent planes, α and β touch the focal surface along curves of the class $2(n-1)$. *These curves of contact are precisely those which correspond to the intersection* $\overline{\alpha\beta}$ (Art. 8).

In proof of this, since the latter curves are of the same class as the former, it will suffice to show that every tangent of the one likewise touches the other. Now, if p touch the curve $\overline{\alpha}^n$, for instance, the curve, in β, corresponding to p will touch $\overline{\alpha\beta}$; so that, of the tangent planes to the focal surface which pass through p, one more than the

* Cremona's *Curve Piane*, Arts. 51*a*.
† Cremona's *Curve Piane*, Art. 52.
‡ *Allgemeine Theorie der geradlinigen Strahlensysteme* (*Journal für die reine und angewandte Mathematik*, Band 57, p. 202).

ordinary number, 2 $(n-1)$, will coincide with a : in other words, p is a tangent to the curve of contact between a and the focal surface.

14. It follows at once, from Art. 7, that the congruence-curves a_{n+1} and b_{n+1} in a and β also lie on the focal surface. But the tangent plane to the latter which passes through an arbitrary tangent p of one of these curves, say a_{n+1}, is in general inclined to the plane a which contains it. For, since every such line p joins a point on $\overline{a\beta}$ to its correspondent in a, the curve in β which corresponds to it will necessarily *cut* $\overline{a\beta}$ at the same point as p itself does, and the plane through p which touches that curve at $(p, \overline{a\beta})$ is clearly the tangent plane in question.

15. It may further be observed that, since every point of a situated on the focal surface must lie either on the curve of contact $\overline{a}^n$ or on the congruence-curve a_{n+1}, the section of this surface made by the plane a must consist of the former curve, counted twice, and of the latter one, which is in general of the order $2n$, counted once ; in other words, *the order of the focal surface of a Cremonian Congruence is, in general*, $4n$, —an inference which is in accordance with the well-known rule, that the difference between the order and the class of any congruence is half the difference between the order and the class of its focal surface.*

16. The above results become modified, in various ways, when special positions are given to the two planes between which a general Cremonian correspondence of the degree n has been established. Of these modifications, a few of the more noteworthy shall now be briefly considered.

17. If two corresponding points coincide in C, on the intersection $\overline{a\beta}$, one of the congruence-rays proceeding from every point in space will obviously pass through it ; so that the congruence $(n+2, n)$ will break up into the congruence (1, 0), consisting of all rays through C, and a residual congruence $(n+1, n)$ of which C is a singular point. This self-corresponding point C, in fact, becomes the centre of a pencil of congruence-rays situated in a singular plane γ ; for, of the n congruence-rays situated in an arbitrary plane passing through C, it is clear that one, and only one, will always pass through this point.

18. Since the existence of a self-corresponding point C diminishes, by one, the order of the Cremonian Congruence, the class of each of the congruence-curves in a and β must suffer a like reduction (Art. 7). The intersection $\overline{a\beta}$ is a $(n-1)$-ple tangent of these congruence-

* This theorem is attributed by Sophus Lie (*Göttinger Nachrichten*, 1870, No. 4) to Felix Klein ; see also Schubert's *Abzählende Geometrie*, p. 64.

curves a_n and b_n, and both obviously touch the singular plane γ. The curves $\overline{a}^n$ and $\overline{b}^n$, corresponding to $\overline{a\beta}$, both pass through C; their remaining intersections with $\overline{a\beta}$ being, as before (Art. 8), the points of contact, with $\overline{a\beta}$, of the congruence-curves b_n and a_n, respectively.

19. The processes of Arts. 10 and 12 being unaltered by the hypothetical presence of a self-corresponding point C, the class $4(n-1)$ of the focal surface, and the multiplicity $2(n-1)$ of the tangent planes α and β of this surface, remain unchanged. By virtue of the general relation referred to in Art. 15, however, the order of the focal surface is reduced, by two, to $4n-2$.

This surface, moreover, acquires a node (double point) at C. For, if a be any line in α, passing through C, and b the tangent, at C, to the curve in β which corresponds to a, *two* of the n congruence-rays in the plane (ab) will coincide with the intersection of the latter and the singular plane γ; in other words, (ab) and γ are the two focal planes of the congruence-ray $\overline{\gamma, (ab)}$, C being the point of contact of the former with the focal surface. But a and b are obviously corresponding rays of two projective pencils in the planes α and β respectively, so that (ab) envelopes a quadric cone—the cone of contact, in fact, at the node C, of the focal surface. γ is clearly a tangent plane of this cone, and the latter likewise touches α and β along the tangents, at C, to the curves $\overline{a}^n$ and $\overline{b}^n$ which correspond to $\overline{a\beta}$ (Art. 8).

20. The correspondence being still perfectly general, the planes α and β may always be placed so that two self-corresponding points, C and D, will present themselves. The residual congruence (n, n), then generated, is of the order as well as of the class n, and the congruence-curves in α and β are each of the class $n-1$, $\overline{a\beta}$ being a common $(n-2)$-ple tangent thereof. The self-corresponding points C and D are the centres of congruence pencils in fixed singular planes γ and δ, respectively. The order of the focal surface is reduced to equality with its class, which is $4(n-1)$, as before. Of this surface, C and D are double points, and γ and δ double tangent planes; whilst the multiplicity of α and β, as tangent planes thereof, is $2(n-1)$, as before.

21. A noteworthy special case, where two self-corresponding points C and D present themselves, may occur if $\overline{a\beta}$ becomes a self-corresponding line; that is to say, a line upon which the correspondent of each of its own points is situated. This can only happen, of course, when there are principal points, of each of the planes α and β, situated on $\overline{a\beta}$, whose joint-multiplicities amount to $n-1$. Each of the singular planes γ and δ, containing congruence-pencils whose centres are C and D, will in such a case pass through the line $\overline{a\beta}$, and the quadric cones which touch the focal surface at its double points C and D

(Art. 19) will each degenerate to a line-pair, one element of which will be $\overline{\alpha\beta}$ itself.

22. The case, however, with which we shall be most concerned in the sequel is still more special than the one briefly indicated in the last Article. It is that in which more than two, and therefore all points of $\overline{\alpha\beta}$ are self-corresponding ones. Every line incident with $\overline{\alpha\beta}$ may then be regarded as a line passing through two corresponding points, and, if the linear complex consisting of all such lines be detached from the aggregate possessing the property in question, a congruence of the class $n-1$ will remain.

The order of this residual congruence will, of course, be greater by unity than the number of rays, passing through an arbitrary point of β (say), which join two corresponding, but not coincident, points situated in that plane. Now, under the present hypothesis, the congruence-rays in β must consist exclusively of rays belonging to the pencils whose centres are the several principal points of α situated on $\alpha\beta$, and each such ray will have to be counted as often as it intersects the principal curve corresponding to the centre from which it proceeds, at a point not coincident with that centre. But, since every point of $\overline{\alpha\beta}$ is supposed to be self-correspondent, every such principal curve passes once, and in general only once, through its corresponding principal point; so that if $\overline{A}_i$ be any one of the latter, every ray of the pencil $(\overline{A}_i, \beta)$ will be a $(i-1)$-ple congruence-ray, and the total number of such rays passing through an arbitrary point of β will be $\Sigma\,(i-1)\,\overline{a}_i$, where $\overline{a}_i$ denotes the number of principal points $\overline{A}_i$, of the order i, situated on $\overline{\alpha\beta}$, and the summation is to be extended from $i=1$ to $i=n-1$. From this, and the fact that the joint multiplicities of all points $\overline{A}_i$ is, by the hypothesis of Art. 21, $\Sigma i\overline{a}_i = n-1$,

we at once deduce, for the required order of our congruence, the expression* $n-\Sigma\overline{a}_i$.

But another expression for this same order is, obviously, $n-\Sigma\overline{\beta}_i$, where $\overline{\beta}_i$ is the number of principal points of the order i, belonging to the plane β, which are situated on $\overline{\alpha\beta}$. Hence we infer that *when every point of the intersection of the planes α and β is self-correspondent, the number,*

$$\Sigma\overline{a}_i = \Sigma\overline{\beta}_i = m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

of principal points situated on their intersection $\overline{\alpha\beta}$ is the same for both planes, and the order of the residual congruence which is generated is less, by this number m, than the degree n of the Cremonian correspondence existing between the two planes. Its class, as already stated, is $n-1$.

* This expression may be readily verified by the more general method indicated in Art. 6.

23. By the method adopted in Art. 10, it can readily be shown that the focal surface of the congruence $(n-m, n-1)$, under consideration, is of the class $4n-2m-6$, and of the order $4(n-m-1)$. With each of the generating planes α and β coincide $2(n-m-1)$ of the tangent planes which can be drawn to the surface through an arbitrary line in that plane.

Congruences of the First Class.

24. Proceeding to the applications of the foregoing general results, I observe that congruences of the first class are generated by the method of Art. 3 when $n=1$; that is to say, when the correspondence between the planes α and β is homographic.* Such a congruence is, in general, of the third order (Art. 6). Its focal surface, which is of the class 0 (Art. 10), that is to say developable, and of the order 4 (Art. 15), touches the planes α and β along the right lines $\bar{a}$, $\bar{b}$, which correspond to $\overline{\alpha\beta}$ (Art. 13), and cuts them in the congruence-conics, a_2 and b_2, which are enveloped by the congruence-rays joining the points of $\overline{\alpha\beta}$ to their correspondents on $\bar{a}$ and $\bar{b}$, respectively (Art. 15).

Every tangent plane of this focal surface is a singular plane of the congruence; for the latter is only of the first class, whereas the plane in question contains, by hypothesis, two, and actually an infinite number of congruence-rays. In virtue of the last-mentioned property, the plane under consideration necessarily intersects α and β in a pair of corresponding lines. Conversely, every plane containing two corresponding lines of α and β must be a singular one; the congruence-curve in it being the conic which touches these lines at their intersections with $\bar{a}$ and $\bar{b}$. The two lines, in fact, are tangents, respectively, to a_2 and b_2, and the plane which contains them, since it touches two conics having a common tangent $\overline{\alpha\beta}$, envelopes, by a well-known theorem, a developable of the third class. Every two such planes obviously intersect in a congruence-ray; the aggregate of all such intersections, in fact, is precisely the congruence (3, 1) under investigation. When the two planes are consecutive ones, the congruence-ray in which they intersect, and which joins corresponding points of the conics a_2 and b_2, becomes a generator of the developable focal surface, and consequently a tangent to the skew cubic which

* In 1832, at the end of his *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten*, Steiner proposed this question: "To what law are the lines subjected which join the corresponding points of two projective (homographic) planes, or what curved surface is touched by them?" (Steiner's *Gesammelte Werke*, Bd. 1, p. 453.) The second part of the *Abhängigkeit*, however, in which this and similar problems were to have been treated, was never published, and the question, or rather its correlative, appears to have been first solved by Reye, in 1870, in the *Journal für die reine und angewandte Mathematik*, Bd. 74, p. 1.

forms its edge of regression. The three congruence-rays which proceed from any point of this skew cubic coincide obviously with the tangent to the curve at that point.

25. When the homographic planes α and β have a self-corresponding point C, the congruence of the first class which they generate is only of the second order (Art. 17). The point C, however, becomes the centre of a pencil of congruence-rays situated in a plane γ, which passes through the points A and B towards which all the congruence-rays in the planes α and β, respectively, now converge (Art. 18). The focal surface is developable, as before; but it is now of the second order, and has a node at C (Art. 19). It not only touches α and β along the lines $\bar{a}$ and $\bar{b}$, through C, which correspond to $\overline{\alpha\beta}$, but by Art. 14 also cuts these planes in the points A and B. These results are most readily explained by again regarding the focal surface as the envelope of planes, each containing a pair of corresponding lines. There are now, in fact, two distinct series of such planes, since there are two different modes of grouping the several rays of the congruence. The planes of one of these series all pass through C, and envelope a quadric cone which touches α and β in the manner above described; the planes of the other series form a pencil whose axis intersects α and β in A and B. The congruence, consequently, has a *focal line*, as well as a *focal cone*. Each plane through the former contains a congruence-conic which lies on the latter; whilst each tangent plane of the latter contains a congruence-pencil whose centre is on the former. The conic in the plane (ABC), which is common to both series, degenerates to a point-pair (C, D), where D is the point in which the focal line and focal cone touch each other. If the two latter be regarded as constituting a developable of the third class, the congruence (2, 1) under consideration will not be, as before, the aggregate of the intersections of *all* its pairs of planes, but only of those which belong to different series; in other words, the congruence (2, 1) consists of all right lines which touch the focal cone, and are at the same time incident with the focal line.

26. When, finally, the homographic planes α and β have two self-corresponding points C and D, and therefore a self-corresponding line $\overline{\alpha\beta}$, they generate a congruence of the first order, as well as of the first class (Art. 20). The focal surface is now of the order, as well as of the class 0, and the points C and D upon it are of such a kind that at each there are an infinite number of tangent planes, all intersecting in the same line (Art. 21). From this, and the fact that every such tangent plane is a singular plane of the congruence (Art. 24), we may at once infer the existence of two focal lines c and d; the former passing through C, and the latter through D. Each, in fact, is the

axis of a pencil of planes, whereof each constituent contains a pencil of congruence-rays, since it intersects α and β in a pair of corresponding lines having a self-corresponding point. The congruence (1, 1) under consideration is thus resolved, in two different ways, into a series of pencils whose planes pass through one focal line, while their centres lie on the other. It is, in short, the aggregate of all the intersections of planes belonging to different pencils, or, if we please, of all right lines which are incident with *both* the focal lines c and d.*

27. A congruence of the first order and first class is also generated, by the method described in Art. 22, when $n=2$; that is to say, when the planes α and β are in quadric correspondence. Two, associated, principal points thereof are necessarily situated on $\overline{\alpha\beta}$; but if A_1' and A_1'' be the two principal points of α which are not situated thereon, and if B_1' and B_1'' be the principal points of β respectively associated with them, it is clear that every plane through $\overline{A_1'B_1'}$, as well as every plane through $\overline{A_1''B_1''}$, will intersect α and β in a pair of corresponding lines having a self-corresponding point, and will therefore contain a pencil of congruence-rays. The axes $\overline{A_1'B_1'}$ and $\overline{A_1''B_1''}$ of the two pencils of planes are, obviously, the focal lines of the congruence (1, 1) under consideration.†

Congruences of the First Order.

28. Cremonian Congruences of the first order are more numerous than those of the first class. They can all be generated by the method described in Art. 22. To this end, it is essential that each of the planes α and β, between which a correspondence of the degree n exists, should possess $n-1$ principal points situated on their intersection $\overline{\alpha\beta}$, and, since the sum of their multiplicities must also be equal to $n-1$, that they should all be principal single points. This condition being satisfied, and the planes α and β appropriately placed, the focal surface of the generated congruence $(1, n-1)$ will be of the class $2(n-2)$ and of the order 0 (Art. 23); that is to say, it will degenerate to a focal curve of the class $2(n-2)$.

29. The above condition can be satisfied (see Art. 30), for all values of

* The above generation of the congruence (1, 1) was given, in 1868, by Reye, in a paper published in the *Journal für die reine und angewandte Mathematik*, Bd. 69, p. 365.

† The theorem converse to the one on which this mode of generation is based, viz., that *the several rays of a congruence* (1, 1) *determine a quadric correspondence between two arbitrary planes*, is in reality Steiner's,—at least, in the case where the focal lines are real. He founded upon it, in fact, the well-known mode of transformation to which he gave the name of *skew* (schief) *projection* (*Gesammelte Werke*, Bd. 1, p. 407). Reye also gives the theorem in the paper quoted in the preceding note.

$n>1$,* by a correspondence of the type, first studied by De Jonquières,† which is characterised by having, in each plane, a principal point of the order $n-1$ of multiplicity, and consequently $2(n-1)$ principal single points. When $(n-1)$ of the latter are situated on the intersection of both planes, the curves in α and β, of the order n, corresponding to $\overline{\alpha\beta}$ will each break up into this line itself, and the $n-1$ principal lines which correspond, respectively, to the above-mentioned principal points. They connect the latter points, in fact, with the principal multiple point, and each contains one of the principal single points of its own plane which are not situated on $\overline{\alpha\beta}$.

Since every plane π, passing through the principal multiple points A_{n-1} and B_{n-1}, cuts the planes α and β in a pair of corresponding lines, which intersect in a self-corresponding point, it necessarily contains a pencil (P, π) of congruence-rays. *The congruence* $(1, n-1)$ *under consideration*, being the aggregate of all such pencils, *has, consequently,* $f \equiv \overline{A_{n-1}B_{n-1}}$ *for a focal line, and the locus of* P *for a focal curve.* The only points in which the latter cuts either of the planes α and β being the principal points, not situated on $\overline{\alpha\beta}$, it is clearly a curve f^{n-1} of the order $n-1$; moreover, since each plane π contains but one point P, this focal curve must be incident at $n-2$ points with the focal line f. Each of the $n-2$ planes π in which this incidence occurs touches, in fact, at A_{n-1} and B_{n-1} a pair of associated branches of the principal curves corresponding, respectively, to B_{n-1} and A_{n-1}.

30. The congruence $(1, n-1)$ above generated is identical with the one described by Kummer in his *Algebraische Strahlensysteme*,‡ and it may readily be shown, conversely, that every congruence which has a focal line and a focal curve, of the order $n-1$, incident at $n-2$ points with that focal line, will determine a correspondence of the n^{th} degree, and of the Jonquièrian type (Art. 29), between two arbitrary planes α and β. Every point of the intersection $\overline{\alpha\beta}$, moreover, will be self-correspondent, and $n-1$ principal single points of each plane will necessarily be situated thereon. The disposition of principal points predicated in Art. 29 is thus realised.

31. It will be observed that when $n=2$, in Art. 29, the congruence $(1, 1)$ generated in Art. 27 is reproduced, and that the value $n=3$ yields a congruence which is precisely the correlative of that described in Art. 25. To generate the congruence $(1, 3)$ correlative to that in Art. 24, however, by the method of Art. 22, a quartic correspondence

---

* When $n=1$, the planes α and β, as is well known, are in perspective with one another; in other words, they generate a congruence $(1, 0)$.

† *Comptes Rendus de l'Académie des Sciences*, tome 49, 1859; and *Nouvelles Annales de Mathématique, deuxième serie*, t. 3, 1864.

‡ *Abhandlungen der K. Acad. der Wissenschaften zu Berlin*, 1866, § 2, V.

between α and β of a different type from that employed in Art. 29 is requisite. In each plane, in fact, there must be three principal double, and three principal single points ;* the latter being all situated on $\overline{\alpha\beta}$.

The principal double points A_2', A_2'', A_2''' of α, and B_2', B_2'', B_2''' of β being arbitrarily placed, let the principal single points A_1', A_1'', A_1''' of α, and B_1', B_1'', B_1''' of β correspond, respectively, to the principal lines $\overline{B_2''B_2'''}$, $\overline{B_2'''B_2'}$, $\overline{B_2'B_2''}$, $\overline{A_2''A_2'''}$, $\overline{A_2'''A_2'}$, $\overline{A_2'A_2''}$, and be situated at the intersections of the latter with $\overline{\alpha\beta}$. Granting for a moment (see Art. 34) the possibility of such a disposition, it is manifest, from Art. 28, that the generated congruence will be of the first order and third class, and that its focal surface will degenerate to a curve of the fourth class. Unlike the congruence (1, 3) generated in Art. 29, however, there is here no focal line, but simply a focal curve f^3 of the third order.

32. To show this, and at the same time to elucidate the relation which exists between the two congruences (1, 3) just referred to, we may proceed as follows:—Let A and B be any two corresponding points of the quartic correspondence in Art. 31. To each conic of the pencil $(A_2'A_2''A_2'''A)$ will correspond, point to point, a conic of the pencil $(B_2'B_2''B_2'''B)$,† and since the two conics will necessarily intersect in two self-corresponding points, the congruence-lines joining their remaining pairs of corresponding points will, as is well known, form a quadric regulus.‡ The aggregate of all such *reguli* is clearly the congruence under consideration; and, since one congruence-ray only passes through an arbitrary point in space, these reguli, or, to speak more strictly, the quadric surfaces on which they lie, form a pencil the base-curve of which breaks up into the congruence-ray $\overline{AB}$, and a cubic which is incident at two points, not only with that ray, but with every other belonging to the congruence. This is the focal cubic f^3; it obviously passes through, and is determined by, the six principal double points of the correspondence.

33. The quadric reguli conjugate, respectively, to those of the pencil considered in the last Article—that is to say, situated with the latter on the several quadric surfaces of the pencil—manifestly form a second congruence of the same order and class as the first one. This

---

* *Bulletin*, p. 216, Art. 20.

† The complete curve, of the eighth order, corresponding to the first of these conics breaks up into the second, and the three principal conics corresponding, solely, to the principal double points A_2', A_2'', A_2''', respectively.

‡ I here adopt a convenient term suggested to me by Prof. Cayley. In the absence of self-corresponding points, the regulus would be of the fourth degree, as Cremona has shown in his *Preliminari*, § 54, p. 43.

new congruence, however, is of the type considered in Art. 29, since each of its rays is clearly incident, at one point solely, not only with the line $\overline{AB}$, but also with the cubic f^3. The number of congruences of this latter kind, which can be similarly associated with a given one of the first kind, is doubly infinite,—seeing that every chord of f^3 may play the part above assigned to $\overline{AB}$.

34. It only remains to be shown, conversely, that a congruence (1, 3) whose rays are chords of any given skew cubic f^3 necessarily determines a correspondence, between two arbitrary planes α and β, of the precise kind predicated in Art. 31.

In the first place, every point of $\overline{\alpha\beta}$ is rendered self-correspondent by the congruence-ray which passes through it. Moreover, the rays of the congruence which are incident with an arbitrary line of either plane, say a in α, generate, as is well known, a quartic scroll upon which f^3 is a double curve, and of which the three congruence-rays situated in α are generators. Every such scroll, therefore, intersects β in a quartic curve, corresponding to a, which possesses three double points, at the points in which f^3 pierces β, and passes necessarily through the intersections of $\overline{\alpha\beta}$ with the above three generators situated in α. The former represent the principal double, the latter the principal single, points of the quartic correspondence employed in Art. 31.

35. All congruences of the first order belong to one of the two types considered in Arts. 29 and 31,* and the method of resolution into quadric reguli employed in Art. 32 may be applied to each of the first three of the series of congruences considered in Art. 29, but to no others. The reguli conjugate to those into which any one of these three congruences may be resolved, form, in the aggregate, a second congruence of the same order and class as the original one. In the case of $n = 4$, these two associated congruences, as we have seen in Art. 33, are of different types; in the two other cases, however, they are of the same type.

When $n = 3$, for example, in Art. 29, every conic of the pencil $(A_2 A_1 A_1' A)$, in α, corresponds to a conic of the pencil $(B_2 B_1'' B_1''' B)$, in β; where A_2, B_2 are the principal double points of the correspondence, A_1, A_1', B_1'', B_1''' the principal single points not situate on $\overline{\alpha\beta}$, and A, B an arbitrary pair of corresponding points. The congruence-rays joining the corresponding points of any two such conics generate, as before, a quadric regulus, and all such reguli, a pencil in which every ray of the congruence (1, 2) under consideration is included. The base of this pencil breaks up into the line $\overline{AB}$ common to all its

* Kummer's *Algebraischen Strahlensysteme*, § 2, IV., and § 2, V.

reguli, the focal line $\overline{A_2B_2}$ of the congruence, and its focal conic which, as we know, is incident with each of the above two lines, and likewise passes through A_1, A_1', B_1'', B_1''' (Art. 29). The congruence (1, 2) formed by the reguli conjugate to those just considered has obviously the same focal conic; but $\overline{AB}$ and $\overline{A_2B_2}$ change places, the former being the focal line, and the latter an ordinary congruence-ray.

Again, when $n = 2$, every conic in α, of the pencil $(A_1A_1'AA')$ corresponds to a conic in β of the pencil $(B_1B_1'BB')$, where A_1, B_1 and A_1', B_1' are the two pairs of associated principal points not situated on $\overline{\alpha\beta}$, and A, B and A', B' are any two pairs of corresponding points. The congruence-rays passing through corresponding points of corresponding conics in these two pencils form a quadric regulus, and all such reguli a pencil in which every ray of the given congruence (1, 1) is included. The base-curve of the pencil obviously breaks up into the focal lines $\overline{A_1B_1}$, and $\overline{A_1'B_1'}$ of that congruence, and into the congruence-rays $\overline{AB}$ and $\overline{A'B'}$ common to every regulus of the pencil. The congruence (1, 1) formed by all the reguli conjugate to the above, is precisely of the same type as the original one; $\overline{A_1B_1}$ and $\overline{A_1'B_1'}$, however, are now ordinary congruence-rays, whilst $\overline{AB}$ and $\overline{A'B'}$ are focal lines.

By giving suitable positions to any two of the four points A, A', B, B' which do not correspond to each other, all the ∞^4 possible modes of resolution into quadric reguli are obtainable.

Resolution of a Congruence into Quadric Reguli.

36. Before proceeding to the detailed consideration of Cremonian Congruences of the second class, or second order, it will be convenient to give a brief, but more general statement of the properties resulting from the resolution of a congruence (m, n) into a one-fold system of quadric reguli.

(**a**) Of the quadric surfaces upon which such reguli are situated, m will pass through an arbitrary point, and n will touch an arbitrary plane; whence it follows that the reguli conjugate to the original ones will form a second, *associated*, congruence (m, n) of the same order and class as the first.

(**b**) Two successive quadric surfaces of the system will intersect in a quartic curve, from every point of which will proceed, in general, two ultimately coincident rays of each of the associated congruences.* The latter, consequently, will, in general, have the envelope of the given system of quadric surfaces for a common focal surface.

* Exceptions to this rule have already presented themselves in the last Article. I do not consider them further.

(**c**) A point P through which every quadric of the system passes must be a singular point of both the associated congruences (m, n). The sum of the orders of the two congruence-cones,* having P for a common vertex, is in all cases n, and between their generators a (1, 1) correspondence exists, corresponding generators being defined to be those which lie on the same quadric surface. The plane of two such generators envelopes the cone of contact with the common focal surface at its node (multiple point) P. The class of this cone cannot exceed n.†

Upon each of the n quadrics, in fact, which touch an arbitrary plane π, passing through P, there will be a line of each of the conjugate reguli situated entirely in π. But only one of these will pass through P, seeing that π, being arbitrary, will not touch at P any one of the n quadrics under consideration. Hence it is the sum of the orders of the two congruence-cones at P which is equal to n.

(**d**) Correlatively, a plane π which is touched by every quadric of the system must be a singular plane of each of the two associated congruences (m, n). The sum of the classes of the two congruence-curves in this plane π is in all cases m, and between their tangents a (1, 1) correspondence exists, provided corresponding tangents be those which are generators of the same quadric. The locus of the intersection of two such tangents is the curve of contact of the common focal surface with its multiple tangent plane π. Its order cannot exceed n.†

(**e**) If the system of quadrics contain a cone, its generators will belong to both systems of conjugate reguli, and therefore be common rays of the two associated congruences (m, n).

(**f**) In like manner, if a quadric surface of the system degenerate to a conic, all its tangents will be common rays of the two associated congruences.

The existence of these degenerate forms (**e**) and (**f**) will, of course, cause a diminution to take place in the class of the cone of contact referred to in (**c**), as well as in the order of the curve of contact to which allusion is made in (**d**).

(**g**) If any quadric surface of the system degenerate to a pair of planes γ, δ, together with a pair of points C, D situated on their intersection (point-and-plane pair), these planes and points will obviously be singular ones of both the associated congruences. One of the latter, in fact, will contain the pair of congruence-pencils (C, γ), (D, δ), and the other the pair (C, δ), (D, γ).

---

* Each may degenerate to a plane pencil.

† A diminution is caused by the presence of coincident corresponding generators, as explained below under (**e**), (**f**), and (**g**).

Congruences of the Second Class.

37. Congruences of the second class, and of any order not exceeding the fourth, may be generated in the manner described in Arts. 3 to 21, provided a quadric correspondence exists between the planes α and β.

The congruence (4, 2) presents itself whenever this correspondence is unconditioned. The intersection $\alpha\beta$ is a double ray thereof, as well as a double tangent of each of the congruence-curves a_3, b_3, of the third class, which α and β contain (Art. 7). The tangents of these curves connect the several points of $\alpha\beta$ with their correspondents on the conics $\overline{a}^2$ and $\overline{b}^2$, and the points in which the former is cut by the latter, respectively, are likewise the points in which $\alpha\beta$ is touched by b_3 and a_3. Moreover, $\overline{a}^2$ and a_3, as well as $\overline{b}^2$ and b_3, have triple contact with each other (Art. 8).*

The focal surface F_4^8 of the congruence is of the order 8 and class 4 (Arts. 10 and 15); it touches the planes α and β at the several points of the conics $\overline{a}^2$ and $\overline{b}^2$ (Art. 13), and cuts them in the congruence-curves a_3 and b_3 (Art. 14).

The conic $\overline{a}^2$ passes through the three principal points A_1', A_1'', A_1''', and $\overline{b}^2$ through their respective associates B_1', B_1'', B_1'''. All six are singular points of the congruence; centres, in fact, of congruence-pencils situated in the planes which connect them with their corresponding principal lines. These six singular planes will be thus denoted:

$$\alpha_1 \equiv (A_1'\overline{B_1''B_1'''}), \quad \beta_1 \equiv (B_1'\overline{A_1''A_1'''}),$$
$$\alpha_2 \equiv (A_1''\overline{B_1'''B_1'}), \quad \beta_2 \equiv (B_1''\overline{A_1'''A_1'}),$$
$$\alpha_3 \equiv (A_1'''\overline{B_1'B_1''}), \quad \beta_3 \equiv (B_1'''\overline{A_1'A_1''}).$$

To a right line in α passing through any principal point corresponds, point by point, a right line passing through the associated principal point in β, and two, but only two, such lines will be incident with their corresponding ones. The congruence-lines which connect the corresponding points of two such incident lines envelope a congruence-conic, touching α and β at points on the conics $\overline{a}^2$ and $\overline{b}^2$. Let the six singular planes thus obtained be

$$\gamma_1 \text{ and } \delta_1, \quad \gamma_2 \text{ and } \delta_2, \quad \text{and} \quad \gamma_3 \text{ and } \delta_3,$$

passing, respectively, through

$$A_1' \text{ and } B_1', \quad A_1'' \text{ and } B_1'', \quad \text{and} \quad A_1''' \text{ and } B_1''';$$

and let the congruence-conics they respectively contain be designated by (γ_1') and (δ_1'), (γ_2') and (δ_2'), and (γ_3') and (δ_3').

38. The congruence (4, 2) above generated, possessing six singular points and fourteen singular planes, is precisely the correlative of the

* Schröter, *Journal für die reine und angewandte Mathematik*, Bd. 54, p. 38.

one described by Kummer in § 9 of his *Algebraische Strahlensysteme*. That it is a perfectly general one, of its order and class, follows at once from the circumstance that any such general congruence must necessarily determine a quadric correspondence between those two of its fourteen singular planes which are distinguished from the rest by containing congruence-curves of the third class. The truth of this statement may be proved in a manner similar to that employed in Art. 34.

39. The congruence (4, 2) under consideration contains three systems of quadric reguli; they proceed from the three pairs of pencils of corresponding lines referred to in Art. 37. Each system includes four degenerate reguli. Each of two of these consists, as we have seen, of the tangents to a conic. Each of the remaining two is represented by a pencil-pair, as shown below; its directrices are principal lines.

Corresponding Directrices.		Pencil-pair.	System.
$\overline{A_1'A_1''}$	$\overline{B_1'B_1'''}$	$(A_1''\alpha_2),\ (B_1'''\beta_3)$	(i.)
$\overline{A_1'A_1'''}$	$\overline{B_1'B_1''}$	$(A_1'''\alpha_3),\ (B_1''\beta_2)$	
$\overline{A_1''A_1'''}$	$\overline{B_1''B_1'}$	$(A_1'''\alpha_3),\ (B_1'\beta_1)$	(ii.)
$\overline{A_1''A_1'}$	$\overline{B_1''B_1'''}$	$(A_1'\alpha_1),\ (B_1'''\beta_3)$	
$\overline{A_1'''A_1'}$	$\overline{B_1'''B_1''}$	$(A_1'\alpha_1),\ (B_1''\beta_2)$	(iii.)
$\overline{A_1'''A_1''}$	$\overline{B_1'''B_1'}$	$(A_1''\alpha_2),\ (B_1'\beta_1)$	

No two reguli of the same system have any generator in common; but two reguli belonging to different systems have always one, and in general only one, common generator.* Should the latter have more than one, however, they will have an infinite number of common generators. The above pencil-pairs present instances of this.

The quadric surfaces which contain the several reguli of one and the same system, all pass through two fixed points and touch eight fixed planes. These points are the centres of the pencils whence the several reguli of the system proceed. Two of the eight planes are obviously α and β; two others connect the centres above referred to with their corresponding principal lines; the remaining four are the planes which contain congruence-conics (Art. 37), but do not pass through either of the last-mentioned centres.

The quadrics of each of the three systems envelope the focal surface

* See end of Art. 45.

of the congruence under consideration (Art. 36**b**), and the reguli upon them which are conjugate to those of which this congruence is the aggregate form another congruence of the same order and class as the original one (Art. 36**a**). The original congruence, and the three thus derived from it, form the four congruences (4, 2) which, as is well known, have a common focal surface, F_4^8, of the eighth order and fourth class.

In the following Table the original congruence is denoted by **C**, the three associated ones, proceeding respectively from the pairs of projective pencils (A_1'), (B_1'); (A_1''), (B_1''); (A_1'''), (B_1'''), by $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$. To each a column is appropriated in which the nature of the congruence-curve is indicated which lies in each of the fourteen singular planes. To each of the latter a separate line is devoted. A congruence-pencil is denoted by the letter indicative of its centre; congruence-conics, by the symbols already used in Art. 37, and other similar ones; and congruence-curves of the third class by the number (3), simply. The theorems of Art. 36 suffice, in all cases, for the verification of these indications.

	C	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$
α	(3)	A_1'	A_1''	A_1'''
β	(3)	B_1'	B_1''	B_1'''
α_1	A_1'	(3)	B_1'''	B_1''
β_1	B_1'	(3)	A_1'''	A_1''
α_2	A_1''	B_1'''	(3)	B_1'
β_2	B_1''	A_1'''	(3)	A_1'
α_3	A_1'''	B_1''	B_1'	(3)
β_3	B_1'''	A_1''	A_1'	(3)
γ_1	(γ_1')	(γ_1')	(γ_1'')	(γ_1'')
δ_1	(δ_1')	(δ_1')	(δ_1'')	(δ_1'')
γ_2	(γ_2')	(γ_2'')	(γ_2')	(γ_2'')
δ_2	(δ_2')	(δ_2'')	(δ_2')	(δ_2'')
γ_3	(γ_3')	(γ_3'')	(γ_3'')	(γ_3')
δ_3	(δ_3')	(δ_3'')	(δ_3'')	(δ_3')

40. The three congruences $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ determine quadric correspondences between the pairs of planes α_1, β_1; α_2, β_2; α_3, β_3, respectively, and from each congruence the other two, as well as **C** itself, may be derived in exactly the same manner as they were themselves derived from **C**. Hence it readily follows that the rays of any one of the possible six *pairs* of congruences may be so grouped as to form conju-

gate reguli on a system of quadric surfaces, all passing through two points, touching eight planes, and enveloping the common focal surface F_4^8. Although the properties of these systems are discernible in the Table of the last Article, they may be thus enunciated :—

Every two of the four associated congruences (4, 2) have the tangents of two conics for common rays. These conics are situated in the planes which pass through the two singular points that are common to every quadric surface of the system originating from the two congruences in question.* Both conics, as well as every quadric surface of the system to which (as degenerate forms) they belong, touch the remaining four planes containing congruence-conics. The four other planes which are touched by every quadric of the system under consideration are those which contain the congruence-curves, of the third class, belonging to the two congruences whence the system proceeds. The four singular planes not yet referred to are those of the two plane-and-point pairs included in the systems of quadrics. The mode in which they are paired is obvious from the points they contain.

41. The fact that the quadrics of each of the above six systems envelope the common focal surface of the four congruences (4, 2), enables us readily to determine the singularities of that surface.

Each of the six singular points, for instance, is a double point thereof; the quadric cone of contact thereat is touched by each of the six singular planes which pass through the point in question. Through this point, in fact, every quadric surface of each of two systems passes,† and in each of these systems the corresponding generators, through the point, are corresponding rays of two plane pencils (Art. 36 **c**). The cone of contact above referred to is obviously the envelope of the plane of two such rays.

Each of the fourteen singular planes, again, is a double tangent-plane of the focal surface, touching the latter along a conic.

Each of the first eight singular planes in the Table of Art. 39, for example, is touched by every quadric of each of three different systems ; and of the corresponding generators in it (relative to any one of these systems) one is always a tangent to the third class congruence-curve situated in the plane, and the other a ray of a pencil. Seeing that two self-corresponding generators exist, viz. : those belonging to the two conics included in any one of the three systems under consideration (Art. 39), the locus of the intersection of corresponding generators is a conic (Art. 36 **d**). This is the conic of contact between the plane under consideration and

* The same two planes also contain the two conics whose tangents are rays common to the remaining two of the four associated congruences.

† That there are *two* systems, follows readily from the preceding note.

the focal surface. It passes, obviously, through the three singular points situated in the plane.

Each of the last six singular planes in the Table of Art. 39, on the other hand, is touched by every quadric of each of four systems, and (relative to any one of these systems) the pairs of corresponding generators in the plane are tangents of the two different congruence-conics which it contains. Here, as before, two self-corresponding generators have to be taken into account; so that the locus of the intersection of the remaining pairs of corresponding generators is again a conic; viz., the conic of contact between the plane under consideration and the focal surface. This conic passes through the two singular points situated in its plane, because each of the latter belongs to one of the two plane-and-point pairs included in any one of the four systems of quadrics above alluded to.

42. The correspondence between α and β being still quadric, as in Art. 37, we will now consider the effect of the presence of one self-corresponding point.

It follows from Arts. 17 to 19, where the general effect of such a point was described, that the congruence (3, 2) now generated, although still of the second class, is only of the third order; that the intersection $\overline{\alpha\beta}$, instead of being a double, is now only a single congruence-ray; that the congruence-curves in α and β which touch it, instead of being of the third, are only of the second class; and further that, with the self-conjugate point C, an additional, fifteenth, singular plane γ is introduced containing a congruence-pencil whose centre is C.

The focal surface F_4^6 moreover, although still of the fourth class, is now only of the sixth order; it cuts the planes α and β, in fact, in the congruence-conics they contain, and touches them at all points of the conics, $\overline{a^2}$ and $\overline{b^2}$, corresponding to $\overline{\alpha\beta}$. The last-mentioned conics, it may be observed, pass through C; each has double contact with the congruence-conic in its own plane, and cuts $\overline{\alpha\beta}$ a second time in the point at which this line is touched by the congruence-conic in the other plane (Art. 8).

43. The above, however, are far from being the only modifications, in preceding results, due to the introduction of the self-corresponding point C. Through the latter, for example, three of the six planes pass which formerly contained congruence-conics; and these three, say γ_1, γ_2, γ_3, now contain congruence-pencils whose respective centres C_1, C_2, C_3 constitute, with C, four additional singular points of the congruence, thus raising the total number from six to ten. Again, since each of the three last-mentioned planes obviously intersects γ in a congruence-line, the centres C_1, C_2, C_3 must all lie, with C, in γ. For a similar reason, the pairs of planes α_1, β_1; α_2, β_2; and α_3, β_3, containing

congruence-pencils whose respective centres are A_1', B_1'; A_1'', B_1''; and A_1''', B_1''' (Art. 37), must pass, in pairs, through C_1, C_2, and C_3, respectively. Finally, the planes δ_1, δ_2, δ_3, each of which obviously intersects, in congruence-rays, the two planes of the group γ_1, γ_2, γ_3 whose suffixes are unlike its own, must pass, respectively, through the pairs of points C_2, C_3; C_3, C_1; C_1, C_2. In short, four of the ten singular points of the congruence **C**, under consideration, must lie in each of its fifteen singular planes. The Table of Art. 49 shows, at once, what points lie in each of these planes.

44. The congruence **C** determines a quadric correspondence, not only between the planes α and β, by which it was generated, but likewise between any two of the five planes α, β, δ_1, δ_2, δ_3 containing congruence-conics, and these two have always a self-corresponding point. The proof of the first of these statements is similar to that given in Art. 34; the correctness of the second follows immediately from the fact that, in the congruence **C**, the point common to any two of the five planes in question is always the centre of a congruence-pencil in a third plane. It may further be observed that associated principal points of the quadric correspondence between the two planes under consideration always lie in one and the same column of the Table just referred to.

45. The congruence **C** contains five systems of quadric reguli. Three of these proceed, as in Art. 39, from the pencils of corresponding lines, in α and β, whose centres are the associated principal points A_1', B_1'; A_1'', B_1''; and A_1''', B_1'''. Of the remaining two, one proceeds from the pencils C (β) and (A_1', A_1'', A_1''', C), whose elements (lines and conics) correspond to each other; the other, in like manner, from the pencils C (α) and (B_1', B_1'', B_1''', C). It is well known, in fact, that if the points of a conic and of a right line (or of another conic), not in the same plane with the former, correspond projectively, the lines connecting their corresponding points form a quadric regulus, provided one point (or two points) be self-correspondent. From this it readily follows, too, that the congruence **C** contains no other quadric reguli than those included in the above five systems.

Each of these systems contains, as before, four degenerate reguli, only one of which, however, now consists of the tangents to a conic, the other three being pencil-pairs. The latter, together with the corresponding elements of the generating pencils which serve as their directrices, are given in the following Table :—

Corresponding Directrices.		Pencil-pair.		System.
$\overline{A_1'A_1''}$	$\overline{B_1'B_1'''}$	$(A_1''\alpha_2)$,	$(B_1'''\beta_3)$	(i.)
$\overline{A_1'A_1'''}$	$\overline{B_1'B_1''}$	$(A_1'''\alpha_3)$,	$(B_1''\beta_2)$	
$\overline{A_1'C}$	$\overline{B_1'C}$	$(C\gamma)$,	$(C_1\gamma_1)$	
$\overline{A_1''A_1'''}$	$\overline{B_1''B_1'}$	$(A_1'''\alpha_3)$,	$(B_1'\beta_1)$	(ii.)
$\overline{A_1''A'}$	$\overline{B_1''B_1'''}$	$(A_1'\alpha_1)$,	$(B_1'''\beta_3)$	
$\overline{A_1''C}$	$\overline{B_1''C}$	$(C\gamma)$,	$(C_2\gamma_2)$	
$\overline{A_1'''A_1'}$	$\overline{B_1'''B_1''}$	$(A_1'\alpha_1)$,	$(B_1''\beta_2)$	(iii.)
$\overline{A_1'''A_1''}$	$\overline{B_1'''B_1'}$	$(A_1''\alpha_2)$,	$(B'\beta_1)$	
$\overline{A_1'''C}$	$\overline{B_1'''C}$	$(C\gamma)$,	$(C_3\gamma_3)$	
$(\overline{CA_1'}, \overline{A_1''A_1'''})$	$\overline{CB_1'}$	$(C_1\gamma_1)$,	$(B_1'\beta_1)$	(iv.)
$(\overline{CA_1''}, \overline{A_1'''A_1'})$	$\overline{CB_1''}$	$(C_2\gamma_2)$,	$(B_1''\beta_2)$	
$(\overline{CA_1'''}, \overline{A_1'A_1''})$	$\overline{CB_1'''}$	$(C_3\gamma_3)$,	$(B_1'''\beta_3)$	
$\overline{CA_1'}$	$(\overline{CB_1'}, \overline{B_1''B_1'''})$	$(C_1\gamma_1)$,	$(A_1'\alpha_1)$	(v.)
$\overline{CA_1''}$	$(\overline{CB_1''}, \overline{B_1'''B_1'})$	$(C_2\gamma_2)$,	$(A_1''\alpha_2)$	
$\overline{CA_1'''}$	$(\overline{CB_1'''}, \overline{B_1'B_1''})$	$(C_3\gamma_3)$,	$(A_1'''\alpha_3)$	

The relation (Art. 39) also subsists, in virtue of which no two reguli of the same system have any generator in common, whilst two reguli belonging to different systems have, in general, one, and only one, common generator. This relation, in fact, is due to the circumstance that to an intersection, not at a principal point, of any two directrices situated in one of the generating planes, necessarily corresponds an intersection of their corresponding directrices in the other plane.

46. The reguli conjugate to those of any one of the above five systems form, in the aggregate, another congruence (3, 2) having the same singular points and planes as $\mathbf{C}$, and the same focal surface F_4^6; the latter being, in fact, the envelope of the quadric surfaces upon which the several pairs of conjugate reguli of the system are situated.

The five congruences thus associated with $\mathbf{C}$, are indicated by the symbols $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$, $\mathbf{C}_5$ in the Table of Art. 49, where a column is reserved wherein to record the singularities of each; all these singularities, it may be observed, are readily deducible from the theorems of Art. 36. Each of these five congruences determines, as $\mathbf{C}$ does, a quadric correspondence between any two of its five singular planes

containing congruence-conics. The associated principal points thereof, as well as the self-corresponding point, are found from the Table in the manner already explained in Art. 44. From any one of these congruences, moreover, all the others, as well as **C**, may be derived in precisely the same manner as they were themselves derived from **C** in Art. 45; whence it follows that the rays of any two of the six associated congruences group themselves into conjugate reguli, situated on a system of quadric surfaces, enveloping their common focal surface F_4^6, passing through four fixed singular points, and touching eight fixed singular planes. Each system includes four degenerate quadrics; viz., one doubled plane (bounded by a conic) and three plane-and-point pairs.

47. Any two of the six associated congruences being given, the doubled plane of the system of quadrics proceeding from them is at once recognised in the Table of Art. 49, as that which contains their common congruence-conic. The four singular points situated in this plane are those through which every quadric of the system passes. The three point-and-plane pairs of the system are recognised with equal facility. The *points* of each pair, in fact, appear twice in the columns appropriated to the two congruences; viz., once on each of the lines appropriated to the *planes* of the pair. The eight singular planes, none of which forms a constituent part of any one of the above four degenerate quadrics, are those which are touched by every quadric of the system to which the latter belong.

48. The singularities of the common focal surface F_4^6 of the six associated congruences, regarded as the envelope of the quadric surfaces belonging to any one of the fifteen systems above referred to, may be investigated by aid of the theorems of Art. 36 in the manner described in Art. 41. The results at which we thus arrive are well known, viz.:—

First,—Each of the ten singular points of the configuration is a double point of the focal surface, the quadric cone of contact at which touches the six singular planes which pass through that point.

Secondly,—Each of the fifteen singular planes of the configuration is a double plane of the focal surface, the curve of contact therewith being a conic passing through the four singular points situated in that plane, and having double contact with the congruence-conic it contains. It is in the latter conic that the focal surface cuts the plane in question.

49. The following Table, to which reference has already been frequently made, is arranged like the one at the end of Art. 39. Since

no ambiguity could arise from so doing, however, the congruence-conics of the several planes are here indicated by the symbol (2), simply.

	$\mathbf{C}$	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_4$	$\mathbf{C}_5$
α	(2)	A_1'	A_1''	A_1'''	(2)	C
β	(2)	B_1'	B_1''	B_1'''	C	(2)
α_1	A_1'	(2)	B_1'''	B_1''	(2)	C_1
β_1	B_1	(2)	A_1'''	A_1''	C_1	(2)
α_2	A_1''	B_1'''	(2)	B_1'	(2)	C_2
β_2	B_1''	A_1'''	(2)	A_1'	C_2	(2)
α_3	A_1'''	B_1''	B_1'	(2)	(2)	C_3
β_3	B_1'''	A_1''	A_1'	(2)	C_3	(2)
γ_1	C_1	C	(2)	(2)	B_1'	A_1'
δ_1	(2)	(2)	C_3	C_2	A_1'	B_1'
γ_2	C_2	(2)	C	(2)	B_1''	A_1''
δ_2	(2)	C_3	(2)	C_1	A_1''	B_1''
γ_3	C_3	(2)	(2)	C	B_1'''	A_1'''
δ_3	(2)	C_2	C_1	(2)	A_1'''	B_1'''
γ	C	C_1	C_2	C_3	(2)	(2)

50. I pass, finally, to the case where two self-corresponding points of the quadric correspondence between α and β present themselves. Since they, together with five pairs of arbitrarily chosen corresponding points, suffice to determine the correspondence,* the possibility of their co-existence is, of course, manifest. Moreover, from what has been already explained in Art. 42, it follows, at once, that the congruence generated in such a case is not only of the second class, but also of the second order; that the intersection $\overline{\alpha\beta}$ is no longer a ray thereof; that in place of congruence-curves we have congruence-pencils in α and β; and that, with the new self-corresponding point D, a sixteenth singular plane δ is introduced, containing a congruence-pencil of which that point is the centre.

The focal surface F_4^4, moreover, is now of the fourth order, as well as of the fourth class; it does not cut either of the planes α or β, but simply touches them at the several points of the conics $\overline{a}^2$ and $\overline{b}^2$, passing through C and D, which correspond to $\overline{\alpha\beta}$.

* On the Correlation of Two Planes, Art. 47. (*Proceedings of the London Mathematical Society*, Vol. v., 1874.)

51. Since the line $\overline{\alpha\beta}$ intersects each of its corresponding conics in self-corresponding points, the congruence-rays joining the several points of the former with their corresponding ones in α or β, all converge, as is well known, to a fixed point on the conic $\overline{a}^2$ or $\overline{b}^2$. The points A_1 and B_1, thus determined, are new singular points of the generated congruence (2, 2); they are centres, in fact, of congruence-pencils in α and β, respectively.* In addition to them, there are, of course, the three singular points D_1, D_2, D_3 introduced, with the self-corresponding point D, in precisely the same manner as C_1, C_2, C_3 were introduced, with C, in Art. 43.

On the whole, therefore, the congruence **C**, now under consideration, contains sixteen singular points as well as sixteen singular planes; each of the former being the centre of a congruence-pencil situated in one of the latter, as shown in the first two columns of either of the Tables of Art. 59.

52. It has already been shown in (Arts. 42 and 43) how it may be proved that each of the first fifteen singular planes, given in the Table of Art. 49, passes through four singular points, and it is manifest that the sixteenth δ has precisely the same properties as γ. It is, moreover, obvious from the preceding Article that α and β each contain two singular points more than they formerly did. In short, it may be proved, by reasoning precisely similar to that employed in Art. 43, that every singular plane now contains six singular points. These are all indicated on the several lines of Table I. of Art. 59. From the different arrangement of the latter, given in Table II. of Art. 59, the six singular planes which pass through each singular point are rendered manifest, and a further inspection of the two Tables likewise reveals the well-known property of the configuration, in virtue of which each of the $16 \times 15 = 240$ intersections of two singular planes passes through two singular points, and *vice versâ* every line passing through two singular points lies in two singular planes.

53. Any one of the sixteen singular planes of the congruence **C** under consideration being selected, there are, as we have just seen, five others which pass through the centre of the congruence-pencil it contains, or say *its centre*, and ten which do not do so. We may say, indeed, that there are $\frac{16 \times 5}{2} = 40$ *pairs* of singular planes which pass through one

* The conics of each of the pencils $(A_1A_1'A_1''A_1''')$ and $(B_1B_1'B_1''B_1''')$ cut $\overline{\alpha\beta}$ in pairs of points in involution. In each pencil there are three line-pairs, one element of each of which, being a principal line of the correspondence, cuts $\overline{\alpha\beta}$ in a point corresponding to a principal point, and, therefore, in line with it and the centre of the congruence-pencil in its plane. Hence it follows that the line-pairs of the two pencils determine the same points on $\overline{\alpha\beta}$, and that the two involutions are identical.

another's centres, and $\frac{16 \times 10}{2} = 80$ which do not possess this property. Between the planes of each of these 80 pairs, the rays of the congruence **C** determine a quadric correspondence precisely similar to that, between α and β, by which it was generated in Art. 50. Between the planes of each of the above 40 pairs, however, the congruence **C** determines a cubic correspondence disposed in the manner described in Art. 22.

54. In proof of these statements, I observe that the rays of **C** which are incident with an arbitrary line form a quartic scroll passing through all the sixteen singular points of the configuration, and upon which that line, and another not incident therewith, are double ones. When the first line lies in one of the planes of the pair under consideration, the scroll breaks up into the congruence-pencil in that plane, and a cubic scroll (passing through the ten singular points not situated in the plane) on which the first line is single, while the second one is still double, and passes through the centre of the detached pencil.

Now, in the first of the two cases with which we are at present concerned, the cubic scroll intersects the second plane of our pair—one of the 80 of Art. 53—in a congruence-ray, and in a conic which passes through three singular points, and corresponds, point by point, to the arbitrary line in the first plane. In short, the correspondence between the two planes is in this case a quadric one.

In the second case, the intersection of the pair of planes—one of the 40 of Art. 53—is a congruence-ray, and through every point thereof a second ray passes which converts that point into a self-corresponding one. The double line, moreover, of the cubic scroll is incident with the intersection of the two planes at the centre of the first, so that the scroll cuts the second plane in a cubic of which the above centre is a double point, and which passes through the four singular points of the plane not situated on the intersection.

The method of recognising the associated principal points of the quadric, or cubic, correspondence between any pair of planes will be explained in Art. 60.

55. Returning to the quadric correspondence between α and β by means of which the congruence **C** was generated (Art. 50), I observe that the latter still contains the five systems of quadric reguli proceeding from the pairs of pencils described in Art. 45. With each of these five systems, however, another is now *associated* which proceeds, in like manner, from the corresponding elements of two pencils whose base-points are the principal, and the self-corresponding points of α and β which are not included amongst the base-points of the pencils whence the former system of reguli proceeds.

Thus, with the system proceeding from any one of the following five pairs of pencils on the left, is associated the system which proceeds from the pair of pencils indicated on the same line on the right of the page:—

α	β	System.	α	β
(A_1')	(B_1')	(i.)	$(CDA_1''A_1''')$	$(CDB_1''B_1''')$
(A_1'')	(B_1'')	(ii.)	$(CDA_1'''A')$	$(CDB_1'''B_1')$
(A_1''')	(B_1''')	(iii.)	$(CDA_1'A_1'')$	$(CDB_1'B_1'')$
$(CA_1'A_1''A_1''')$	(C)	(iv.)	(D)	$(DB_1'B_1''B_1''')$
(C)	$(CB_1'B_1''B_1''')$	(v.)	$(DA_1'A_1''A_1''')$	(D)

The four degenerate reguli included in each of the systems above referred to are all pencil-pairs. Arranged in associated systems, they are readily seen to be those exhibited in the Table on the next page. The corresponding elements of the pencils whence, as directrices, each such degenerate regulus proceeds, are placed, in juxta-position with the latter, in the extreme right and left-hand columns.

56. All the sixteen congruence-pencils of **C**, it will be observed, are represented by the degenerate reguli included in each of the five pairs of associated systems, and four such pencils are common to each system and every other not associated therewith.

Moreover, since the co-planar directrices of any two reguli belonging to associated systems intersect in two points, the reguli in question have necessarily two generators in common (Art. 45). From this it follows, as a particular case, that the quadric surfaces on which the several reguli of any system are situated all pass through the eight centres, and touch the eight planes of the four pencil-pairs included in the associated system of reguli.

This theorem is likewise true, of course, for each of the four plane-and-point pairs of the system under consideration; whence it immediately follows that the latter, together with the plane-and-point pairs of the associated system, form two conjugate *quadruple groups*;* that is to say, all the sixteen singular points and planes of the configuration are included in them, and each of the two planes of every plane-and-point pair belonging to one of the two groups passes through one of the two points of every plane-and-point pair included in the other, and

* Caporali terms such a group of eight points (planes) an *ottupla* (*Reale Accademia de Lincei*, Vol. ii., 1878).

Corresponding Directrices.		Pencil-Pair.	System.	Pencil-Pair.	Corresponding Directrices.	
$\overline{A_1'A_1''}$	$\overline{B_1'B_1'''}$	$(A_1''a_2)$, $(B_1'''\beta_3)$	(i.)	(A_1a), $(A_1'a_1)$	$(CDA_1''A_1'''A_1')$	$(\overline{CD}, \overline{B_1''B_1'''})$
$\overline{A_1'A_1'''}$	$\overline{B_1'B_1''}$	$(A_1'''a_3)$, $(B_1''\beta_2)$		$(B_1\beta)$, $(B_1'\beta_1)$	$(\overline{CD}, \overline{A_1''A_1'''})$	$(CDB_1''B_1'''B_1')$
$\overline{A_1'C}$	$\overline{B_1'C}$	$(C\gamma)$, $(C_1\gamma_1)$		$(C_2\gamma_2)$, $(D_3\delta_3)$	$(\overline{CA_1''}, \overline{DA_1'''})$	$(\overline{CB_1''}, \overline{DB_1'''})$
$\overline{A_1'D}$	$\overline{B_1'D}$	$(D\delta)$, $(D_1\delta_1)$		$(C_3\gamma_3)$, $(D_2\delta_2)$	$(\overline{CA_1'''}, \overline{DA_1''})$	$(\overline{CB_1'''}, \overline{DB_1''})$
$\overline{A_1''A_1'''}$	$\overline{B_1''B_1'}$	$(A_1'''a_3)$, $(B_1'\beta_1)$	(ii.)	(A_1a), $(A_1''a_2)$	$(CDA_1'''A_1'A_1'')$	$(\overline{CD}, \overline{B_1'''B_1''})$
$\overline{A_1''A_1'}$	$\overline{B_1''B_1'''}$	$(A_1'a_1)$, $(B_1'''\beta_3)$		$(B_1\beta)$, $(B_1''\beta_2)$	$(\overline{CD}, \overline{A_1'''A_1'})$	$(CDB_1'''B_1'B_1'')$
$\overline{A_1''C}$	$\overline{B_1''C}$	$(C\gamma)$, $(C_2\gamma_2)$		$(C_3\gamma_3)$, $(D_1\delta_1)$	$(\overline{CA_1'''}, \overline{DA_1'})$	$(\overline{CB_1'''}, \overline{DB_1'})$
$\overline{A_1''D}$	$\overline{B_1''D}$	$(D\delta)$, $(D_2\delta_2)$		$(C_1\gamma_1)$, $(D_3\delta_3)$	$(\overline{CA_1'}, \overline{DA_1'''})$	$(\overline{CB_1'}, \overline{DB_1'''})$
$\overline{A_1'''A_1'}$	$\overline{B_1'''B_1''}$	$(A_1'a_1)$, $(B_1''\beta_2)$	(iii.)	(A_1a), $(A_1'''a_3)$	$(CDA_1'A_1''A_1''')$	$(\overline{CD}, \overline{B_1'B_1''})$
$\overline{A_1'''A_1''}$	$\overline{B_1'''B_1'}$	$(A_1''a_2)$, $(B_1'\beta_1)$		$(B_1\beta)$, $(B_1'''\beta_3)$	$(\overline{CD}, \overline{A_1'A_1''})$	$(CDB_1'B_1''B_1''')$
$\overline{A_1'''C}$	$\overline{B_1'''C}$	$(C\gamma)$, $(C_3\gamma_3)$		$(C_1\gamma_1)$, $(D_2\delta_2)$	$(\overline{CA_1'}, \overline{DA_1''})$	$(\overline{CB_1'}, \overline{DB_1''})$
$\overline{A_1'''D}$	$\overline{B_1'''D}$	$(D\delta)$, $(D_3\delta_3)$		$(C_2\gamma_2)$, $(D_1\delta_1)$	$(\overline{CA_1''}, \overline{DA_1'})$	$(\overline{CB_1''}, \overline{DB_1'})$
$(CA_1'A_1''A_1'''D)$	$\overline{CD}$	(A_1a), $(D\delta)$	(iv.)	$(C\gamma)$, $(B_1\beta)$	$\overline{DC}$	$(DB_1'B_1''B_1'''C)$
$(\overline{CA_1'}, \overline{A_1''A_1'''})$	$\overline{CB'}$	$(C_1\gamma_1)$, $(B_1'\beta_1)$		$(D_1\delta_1)$, $(A_1'a_1)$	$\overline{DA_1'}$	$(\overline{DB_1'}, \overline{B_1''B_1'''})$
$(\overline{CA_1''}, \overline{A_1'''A'})$	$\overline{CB_1''}$	$(C_2\gamma_2)$, $(B_1''\beta_2)$		$(D_2\delta_2)$, $(A_1''a_2)$	$\overline{DA_1''}$	$(\overline{DB_1''}, \overline{B_1'''B_1'})$
$(\overline{CA_1'''}, \overline{A_1'A_1''})$	$\overline{CB_1'''}$	$(C_3\gamma_3)$, $(B_1'''\beta_3)$		$(D_3\delta_3)$, $(A_1'''a_3)$	$\overline{DA_1'''}$	$(\overline{DB_1'''}, \overline{B_1'B_1''})$
$\overline{CD}$	$(CB_1'B_1''B_1'''D)$	$(B_1\beta)$, $(D\delta)$	(v.)	(A_1a), $(C\gamma)$	$(DA_1'A_1''A_1'''C)$	$\overline{DC}$
$\overline{CA_1'}$	$(\overline{CB_1'}, \overline{B_1''B_1'''})$	$(C_1\gamma_1)$, $(A_1'a_1)$		$(D_1\delta_1)$, $(B_1'\beta_1)$	$(\overline{DA_1'}, \overline{A_1''A_1'''})$	$\overline{DB_1'}$
$\overline{CA_1''}$	$(\overline{CB_1''}, \overline{B_1'''B_1'})$	$(C_2\gamma_2)$, $(A_1''a_2)$		$(D_2\delta_2)$, $(B_1''\beta_2)$	$(\overline{DA_1''}, \overline{A_1'''A_1'})$	$\overline{DB_1''}$
$\overline{CA_1'''}$	$(\overline{CB_1'''}, \overline{B_1'B_1''})$	$(C_3\gamma_3)$, $(A_1'''a_3)$		$(D_3\delta_3)$, $(B_1'''\beta_3)$	$(\overline{DA_1'''}, \overline{A_1'A_1''})$	$\overline{DB_1'''}$

correlatively each of the two points of the former lies in one of the two planes of every point-and-plane pair included in the latter.

It should be added, that here, as before, two reguli belonging to different and non-associated systems have, in general, one, and only one, generator in common; while two reguli belonging to the same system have no common generator (Art. 45).

57. The reguli conjugate to those included in any one of the preceding systems constitute, in the aggregate, another congruence (2, 2), having the same singular points and planes—in fact, the same focal surface F_4^4 as **C** itself (Art. 36). In passing from any system of reguli to the conjugate system, the centres and planes are simply interchanged, not only of the four pencil-pairs which each such system includes (Art. 36 **g**), but also of the four pencil-pairs determined by the quadrics, on which the reguli of the system are situated, around the four pairs of points through which they all pass, and in the four pairs of planes which they all touch (Art. 56). From this it readily follows that the two congruences are identical which are formed by the reguli conjugate, respectively, to those of any two associated systems (Art. 55). From the congruence **C**, therefore, five, and only five, other congruences can be derived by the method under consideration. In the Tables of Art. 59 these are indicated by $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$, $\mathbf{C}_5$, respectively, and in the column appropriated to each will be found, in Table I., the centres of the pencils situated in the several singular planes, and, in Table II., the planes of the pencils which have the several singular points of the configuration for their centres.

58. Here, as in Arts. 40 and 46, it should be observed that any one of the above-mentioned five congruences may be taken for the primitive one, and that from it five others may be derived in precisely the same manner as that above described.* These five, however, are always identical with the five remaining congruences of the original six. Hence we may at once infer that the rays of any two of the six congruences (2, 2) which have a common focal surface group themselves, and that *in two different ways* (Art. 57),† into conjugate reguli

---

* If α and β be retained as generating planes, however, the correspondence between them will be a cubic one when $\mathbf{C}_4$ or $\mathbf{C}_5$ is taken as the primitive congruence (Art. 53), and, in resolving it into associated systems of quadric reguli, obvious modifications will, of course, have to be introduced (see Art. 65).

† The quartic regulus formed by all rays, of any given congruence (2, 2), which are incident with an arbitrary line (directrix), and on which the latter is one of two double lines (Art. 54), breaks up, therefore, into two quadric reguli when the directrix in question is a ray of any one of the five congruences having, with the given one, a common focal surface. These two quadric reguli belong to one of the pairs of associated systems into which the given congruence can be resolved (Art. 55).

situated on a system of quadrics which include, as degenerate forms, the elements of a quadruple group (Art. 56), and every surface of which passes through the eight points, and touches the eight planes of the conjugate quadruple group.

The total number of different systems of quadrics enveloping the focal surface F_1^4 of a given congruence (2, 2) is obviously 30; and this is also the number of different quadruple groups (15 conjugate pairs) which can be formed from the sixteen singular points and sixteen singular planes of the configuration.*

59. The following are the two Tables to which reference has already been made in Arts. 52 and 57. Their construction is substantially the same as that of the Table given in Art. 49, the main difference being that the singular planes now contain congruence-pencils solely, and that perfect reciprocity everywhere prevails.

Table I.

	$\mathbf{C}$	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_4$	$\mathbf{C}_5$
α	A_1	A_1'	A_1''	A_1'''	D	C
β	B_1	B_1'	B_1''	B_1'''	C	D
α_1	A_1'	A_1	B_1'''	B_1''	D_{1_2}	C_1
β_1	B_1'	B_1	A_1'''	A_1''	C_1	D_1
α_2	A_1''	B_1'''	A_1	B_1'	D_2	C_2
β_2	B_1''	A_1'''	B_1	A_1'	C_2	D_2
α_3	A_1'''	B_1''	B_1'	A_1	D_3	C_3
β_3	B_1'''	A_1''	A_1'	B_1	C_3	D_3
γ_3	C_3	D_2	D_1	C	B_1'''	A_1'''
δ_3	D_3	C_2	C_1	D	A_1'''	B_1'''
γ_2	C_2	D_3	C	D_1	B_1''	A_1''
δ_2	D_2	C_3	D	C_1	A_1''	B_1''
γ_1	C_1	C	D_3	D_2	B_1'	A_1
δ_1	D_1	D	C_3	C_2	A_1'	B_1'
γ	C	C_1	C_2	C_3	B_1	A_1
δ	D	D_1	D_2	D_3	A_1	B_1

Table II.

	$\mathbf{C}$	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_4$	$\mathbf{C}_5$
A_1	α	α_1	α_2	α_3	δ	γ
B_1	β	β_1	β_2	β_3	γ	δ
A_1'	α_1	α	β_3	β_2	δ_1	γ_1
B_1'	β_1	β	α_3	α_2	γ_1	δ_1
A_1''	α_2	β_3	α	β_1	δ_2	γ_2
B_1''	β_2	α_3	β	α_1	γ_2	δ_2
A_1'''	α_3	β_2	β_1	α	δ_3	γ_3
B_1'''	β_3	α_2	α_1	β	γ_3	δ_3
C_3	γ_3	δ_2	δ_1	γ	β_3	α_3
D_3	δ_3	γ_2	γ_1	δ	α_3	β_3
C_2	γ_2	δ_3	γ	δ_1	β_2	α_2
D_2	δ_2	γ_3	δ	γ_1	α_2	β_2
C_1	γ_1	γ	δ_3	δ_2	β_1	α_1
D_1	δ_1	δ	γ_3	γ_2	α_1	β_1
C	γ	γ_1	γ_2	γ_3	β	α
D	δ	δ_1	δ_2	δ_3	α	β

60. The nature of the correspondence, determined by any one of the six congruences (2, 2), between any two given planes may be thus

* Caporali, *loc. cit.*

recognised in Table I. If the points in the column devoted to the congruence in question, and on the lines appropriated to the two planes, are those which are common to the latter (Art. 53), the correspondence is cubic; if not, quadric. In the first case, the common points reappear in another column, on exchanged lines, and there indicate the principal double points of the planes to which those lines are respectively appropriated. In each of the remaining four columns a pair of associated principal single points of the two planes will be found. In the second case, each of two of the remaining five columns contains the self-corresponding points of the two planes, and each of the other three a pair of associated principal single points of the quadric correspondence existing between them.

61. Two conjugate quadruple groups (Art. 56) are determined by any one pair of planes (or points) belonging to either of them. In the one case, Table I. (in the other, Table II.) enables us readily to detect all their remaining elements. The methods of doing so being precisely similar in both cases, we will, for the sake of illustration, confine our attention to Table I., where, in each of two easily detected columns, both the points appear which are common to the two given planes. In each of the remaining four columns, and on the lines appropriated to those planes, will be found a pair of points of the quadruple group conjugate to that to which the given pair of planes belongs. The points of each of the four pairs just determined reappear in each of the above-mentioned two columns, and that on the lines appropriated to the two planes which pass through them. From the latter, finally, the quadruple group to which the given pair of planes belong is completed in a precisely similar manner.

62. If the two columns above referred to had been given, the conjugate quadruple groups could have been determined with equal readiness; and thus we should have solved, by inspection, the problem on which depends the complete definition of each of the two associated systems of quadrics whose conjugate reguli constitute, in the aggregate, the two congruences indicated by the given columns (Art. 58).

63. Little need be added with respect to the singularities of the focal surface F_4^4 common to the six congruences (2, 2) indicated in the preceding Tables. Each of the sixteen singular planes of the configuration is a double tangent-plane of the surface, and touches it along the conic which passes through the six singular points of the plane. This conic is the locus of the intersection of corresponding rays (Art. 36 **d**) of the two congruence-pencils determined by the quadrics which touch that plane, and at the same time belong to one or other of the systems

above defined. Each of the sixteen singular points, again, is a double point of F_4^4; the quadric cone of contact thereat being the envelope of the plane of corresponding rays (Art. 36**c**) of the two congruence-pencils determined by the quadrics, belonging to one of the systems already referred to, which pass through that point. This quadric cone manifestly touches each of the six singular planes of the configuration which pass through the singular point in question.

Congruences of the Second Order.

64. All Cremonian Congruences of the second order may be generated by the method given in Art. 22. The formulæ of that Article become, for such congruences,

$$\sum_1^{n-1} i\bar{a}_i = n-1,$$

$$\sum_1^{n-1} \bar{a}_i = n-2,$$

where n denotes the degree of the correspondence between the two generating planes, and $\bar{a}_i$ the number of principal points, belonging to either of them, which are of the order i and are situated on their intersection $\overline{\alpha\beta}$. From these formulæ we may at once deduce

$$\sum_1^{n-1} (i-1)\,\bar{a}_i = 1,$$

and thence infer, first that no principal point whose order of multiplicity is higher than 2 can lie on $\overline{\alpha\beta}$, secondly that one and only one principal double point of each plane must be so situated, and lastly that $n-3$ principal single points of each plane must also lie on $\overline{\alpha\beta}$.

Kummer having shown that the class, $n-1$, of no congruence of the second order can exceed 7,* it follows readily from the above, and from Cremona's tabulated results,† that the only correspondences available for the generation of congruences of the second order, by the method under consideration, are of the degrees 3, 4, 5, and 6, respectively; and, further, that only one correspondence of each of these degrees can be so employed. The generated congruences are, of course, of the classes 2, 3, 4, and 5, respectively. The general properties of the first three of these are simply correlative to those already studied in Arts. 37 to 63; it will suffice, therefore, to direct attention to the special features revealed by the present mode of generation. The congruence (2, 5), on the other hand, will require a somewhat fuller treatment.

65. Of the cubic correspondence between α and β, let C and D be

* *Algebraische Strahlensysteme*, § 6, XXXV.
† *Bulletin*, Arts. 19 to 23.

the principal double points, the former being supposed to belong to β, the latter to α, and both situated on $\overline{\alpha\beta}$. Moreover, let A_1, B_1; A_1', B_1'; A_1'', B_1''; and A_1''', B_1''' be the four pairs of associated principal single points of the correspondence. From Art. 22, and well-known properties, it follows that the principal conics of the two planes intersect one another in C and D, and that each passes through the four principal single points of its plane.*

The principal double point of each plane is the centre of a congruence-pencil situated in the other plane, and each principal single point is the centre of a congruence-pencil whose plane passes through that point and its corresponding principal line. To each of the six lines, moreover, which pass through two principal single points of either plane corresponds, point by point, the line passing through the two non-associated principal single points of the other plane; and, since two such corresponding lines are necessarily incident in a self-corresponding point, the lines joining their remaining pairs of corresponding points necessarily form a congruence-pencil. The sixteen such pencils are thus all accounted for, and the congruence (2, 2) under consideration is clearly identical with that denoted by $\mathbf{C}_5$ in the Tables of Art. 59.

The five pairs of associated systems of quadric reguli into which $\mathbf{C}_5$ may be resolved (Art. 56) proceed from the following pencils of corresponding directrices, and in each system four degenerate reguli (pencil-pairs) are included:

α	β		α	β
(A_1)	$(CB_1'B_1''B_1''')$	$\mathbf{C}$	$(DA_1'A_1''A_1''')$	(B_1)
(A_1')	$(CB_1B_1''B_1''')$	$\mathbf{C}_1$	$(DA_1A_1''A_1''')$	(B_1')
(A_1'')	$(CB_1B_1'B_1''')$	$\mathbf{C}_2$	$(DA_1A_1'A_1''')$	(B_1'')
(A_1''')	$(CB_1B_1'B_1'')$	$\mathbf{C}_3$	$(DA_1A_1'A_1'')$	(B_1''')
(D)	(C)	$\mathbf{C}_4$	$(A_1A_1'A_1''A_1''')$	$(B_1B_1'B_1''B_1''')$

The reguli conjugate to those of any one of the above ten systems (five associated pairs) constitute, in the aggregate, the congruence indicated in the central column on the line upon which stand the pencils of corresponding directrices from which the system in question proceeds.

66. To generate the congruence (2, 3), by the present method, the

* A_1, B_1; A_1', B_1'; A_1'', B_1''; A_1''', B_1'''; C, D; D, C form, in fact, six pairs of corresponding points of two projective ranges on these conics. (See Note to Art. 51.)

quartic correspondence is employed which has three principal double, and three principal single points in each of the planes α and β.* To any principal single point in either of these planes corresponds an indeterminate point in a principal line passing through two principal double points of the other plane. The remaining principal double point of the latter plane will be termed the *associate* of the principal single point in the former, and the principal points of the planes α and β, thus associated, will be indicated by A_2, B_1; A_2', B_1'; A_2'', B_1''; A_1, B_2; A_1', B_2'; A_1'', B_2'', respectively, where the multiplicity of each point is indicated by a suffix.

If A_2 and A_1 be the two principal points of α, which, by Art. 64, must, in the present case, be situated on $\overline{\alpha\beta}$, it is easy to see that their associates B_1 and B_2 will likewise be so situated. Moreover, A_2 and B_2 will lie on the principal conics $(B_2B_2'B_2''B_1'B_1'')$ and $(A_2A_2'A_2''A_1'A_1'')$ which respectively correspond to them, and A_1 and B_1 on their corresponding principal lines $\overline{B_2'B_2''}$ and $\overline{A_2'A_2''}$.

From this it follows at once that, in the congruence **C** now generated, A_2 and B_2 are centres of congruence-pencils situated in β and α, respectively, and that each of the points A_1', A_1'', B_1', B_1'' is the centre of a congruence-pencil in the plane which connects it with its corresponding principal line. These four planes will be denoted by $\alpha_1, \alpha_2, \beta_1, \beta_2$, respectively.

The remaining four congruence-pencils which **C** possesses proceed from the pairs of corresponding lines $\overline{A_2'A_1'}$ and $\overline{B_2'B_1'}$, $\overline{A_2''A_1''}$ and $\overline{B_2''B_1''}$, $\overline{A_2'A_1''}$ and $\overline{B_2''B_1'}$, and $\overline{A_2''A_1'}$ and $\overline{B_2'B_1''}$. The lines of each pair being incident at a self-corresponding point, the congruence-lines connecting their remaining corresponding points form a pencil. These pencils will be denoted by $(C_1\gamma_1)$, $(C_2\gamma_2)$, $(D_1\delta_1)$, and $(D_2\delta_2)$, respectively. The lines $\overline{C_1C_2}$ and $\gamma_1\gamma_2$, as well as $\overline{D_1D_2}$ and $\overline{\delta_1\delta_2}$, will obviously be identical.

The only five remaining singular points of **C** are vertices of quadric congruence-cones. Four of these vertices are at the principal points A_2', A_2'', B_2', B_2'', and the cones to which they belong pass through the principal conics corresponding to these points. The fifth congruence-cone belongs to one of the systems of quadric reguli into which **C** may be resolved (see Art. 68). It intersects α and β in a pair of corresponding conics of the pencils $(A_2'A_2''A_1'A_1'')$ and $(B_2'B_2''B_1'B_1'')$, and its vertex C lies in each of the four singular planes $\alpha_1, \alpha_2, \beta_1, \beta_2$.

67. Each of the ten singular planes of the congruence **C** under consideration passes through six of its fifteen singular points. The latter, which are readily recognised by methods previously employed (Art. 43),

* *Bulletin*, Art. 20.

are indicated on the line appropriated to that plane in the Table of Art. 69. Two of these six points are always vertices of congruence-cones, and the remaining four of congruence-pencils in **C**. The constituents of each of the forty-five *pairs* of planes that can be formed from the above ten planes have two singular points in common. In each of fifteen pairs, these common singular points are centres of congruence-pencils of **C** in the respective planes; but, in each of the remaining thirty, one is the centre of a congruence-pencil of **C** in a plane not coincident with either of those forming the pair, while the other is the vertex of a congruence-cone.

The nature of the correspondence, determined by **C**, between the planes of a pair depends upon the above circumstances. Between the planes of each of the fifteen pairs, the correspondence is a quartic one of precisely the same kind as that from which **C** was generated. Between the planes of each of the thirty pairs, however, the correspondence is a cubic one of a special nature. Two of its four pairs of associated principal single points coincide at the vertex of the congruence-cone, situated on the intersection of the planes of the pair. The other singular point of **C** on this intersection is simply a self-corresponding point of the pair of planes. The principal double points of the cubic correspondence between them are at the remaining two vertices of congruence-cones which the planes contain. Of the three remaining singular points of **C** situated in each of them, two are principal single points of the cubic correspondence, and the third is the point to which all rays converge which connect the several points of the intersection of the pair of planes with their correspondents. This intersection, in fact, must be regarded as a line passing through one, only, of the two principal single points of its plane which are coincident with, or rather consecutive to, one another, on the principal conic of the plane. To it, therefore, will correspond a cubic which breaks up into a principal line, and a conic corresponding, point by point, to the intersection in question, and cutting it in two self-corresponding points. The convergence, at a point on this conic, of the congruence-rays joining its several points to their corresponding ones on the intersection is an immediate consequence (Art. 51).

68. Returning to the quartic correspondence between α and β, by means of which the congruence **C** was generated (Art. 66), I observe that the resolution of the latter into a system of quadric reguli may be effected by the aid of any one of the following five pencils of corresponding directrices:

$$(A_2') \text{ and } (B_2B_2'B_2''B_1'),$$
$$(A_2'') \text{ and } (B_2B_2'B_2''B_1''),$$
$$(A_2A_2'A_2''A_1') \text{ and } (B_2'),$$

$(A_2A_2'A_2''A_1'')$ and (B_2''),

and $(A_2'A_2''A_1'A_1'')$ and $(B_2'B_2''B_1'B_1'')$.

The reguli conjugate to those of each system form, in the aggregate, another congruence (2, 3) having the same focal surface as **C**. They are indicated in the Table (Art. 69) by $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$, $\mathbf{C}_5$, respectively. The rays of any two of the above six congruences group themselves, in fact, into conjugate reguli on a system of quadric surfaces enveloping the common focal surface, passing, individually, through eight fixed points, and touching four fixed planes. Of such systems of quadrics there are, of course, $\frac{6\times 5}{2} = 15$; each includes four degenerate forms, viz., three point-and-plane pairs, and one cone. The vertex of the latter appears, on one of the last five lines of the Table, in each of the columns appropriated to the two congruences whence the system of quadrics proceeds. The remaining eight points in these two columns, and on the last five lines, are those through which every quadric of the system passes. The three point-and-plane pairs of the system are recognised in the Table, by the fact that the points of the pair appear twice in each of the two columns,—viz., once on each of the two lines indicative of the planes of the pair. The remaining four of the first ten lines of the Table are those which are appropriated to the planes touched by every quadric of the system under consideration. They all pass, of course, through the vertex of the quadric cone included in that system.

69. The following Table exhibits, conveniently, the singularities of the six congruences (2, 3) which have the same focal surface. To each congruence, as before, a column is appropriated, in which, on the first ten lines, stand the centres of the congruence-pencils situated in the several planes to which those lines are appropriated. On the last five lines are the vertices of the quadric cones whose generators are rays of the respective congruences.

	C	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_4$	$\mathbf{C}_5$
α	B_2	A_2'	A_2''	A_1''	A_1'	A_2
β	A_2	B_1''	B_1'	B_2'	B_2''	B_2
α_1	A_1'	C_1	D_2	C	B_2	B_2''
β_1	B_1'	C	A_2	C_1	D_1	A_2''
α_2	A_1''	D_1	C_2	B_2	C	B_2'
β_2	B_1''	A_2	C	D_2	C_2	A_2'
γ_1	C_1	A_1'	B_2'	B_1'	A_2'	C_2
γ_2	C_2	B_2''	A_1''	A_2''	B_1''	C_1
δ_1	D_1	A_1''	B_2''	A_2'	B_1'	D_2
δ_2	D_2	B_2'	A_1'	B_1''	A_2''	D_1
	A_2''	A_2''	B_2	C_2	D_2	B_1'
	A_2'	B_2	A_2'	D_1	C_1	B_1''
	B_2''	C_2	D_1	B_2''	A_2	A_1'
	B	D_2	C_1	A_2	B_2'	A_1''
	C	B_1'	B_1''	A_1'	A_1''	C

70. To generate the congruence (2, 4), the quintic correspondence is employed which has, in each of the planes α and β, a principal triple point, as well as three double, and three single principal points.* To each of the latter corresponds a principal line passing through the triple point and one of the former,—a circumstance which at once associates the double and single principal points of the correspondence, and suggests the notation A_2, B_1; A_2', B_1'; A_2'', B_1''; A_1, B_2; A_1', B_2'; A_1'', B_2''. The principal triple points are denoted by A_3 and B_3.

The conditions of Art. 64 will be satisfied by supposing A_2, A_1', A_1'', B_2, B_1', B_1'' to be all situated on the intersection $\alpha\beta$; whence it will follow, as before, that A_2 and B_2 become centres of congruence-pencils situated in the planes β and α, respectively, and A_1 and B_1 centres of pencils situated in the planes α_1 and β_1, which connect these points with the principal lines respectively corresponding thereto. Moreover, since the lines $\overline{A_1A_3}$ and $\overline{B_1B_3}$ correspond, point by point, to $B_2'B_2''$ and $A_2'A_2''$, respectively, and each is incident with its corresponding line in a self-corresponding point, we have obviously two other congruence-pencils which may be denoted, respectively, by (C, γ) and (D, δ).

In addition to the above singularities, the congruence **C** now under

* *Bulletin*, Art. 21.

consideration possesses six quadric congruence-cones. Four of these have the vertices A_2', A_2'', B_2', B_2'', and pass, respectively, through the principal conics corresponding to these points. Each of the remaining two intersects α and β in a pair of corresponding conics of the pencils $(A_3A_2'A_2''A_1)$ and $(B_3B_2'B_2''B_1)$. In fact, the several pairs of corresponding conics of these pencils are, as will be seen in Art. 72, the directrices of one of the systems of quadric reguli into which the congruence **C** may be resolved. The generators of those reguli which pass through A_1 and B_1 are corresponding rays of two projective pencils $(A_1\alpha_1)$ and $(B_1\beta_1)$; and the double points, C' and C'', of the projective rows which these pencils determine on $\overline{\alpha_1\beta_1}$, are obviously the vertices of the two congruence-cones above referred to.

Lastly, **C** contains two cubic congruence-cones; their vertices are at A_3 and B_3, and their traces on β and α, respectively, are the principal cubics corresponding to these vertices. $\overline{A_3B_3}$ is a double generator of each of these cones, and likewise a double ray of the congruence.

71. Each of the six singular planes of **C** contains six of its fourteen singular points, and to each *pair* of singular planes two singular points are common (see Table of Art. 72). In six of the fifteen different pairs formed by these planes the common points are centres of congruence-pencils situated in those planes; in three others they are both vertices of quadric congruence-cones; and in the six remaining pairs of planes one of the common points is the vertex of a cubic congruence-cone, and the other the centre of a congruence-pencil situated in a plane not coincident with either of those forming the pair.

Between the planes of each of the first six pairs, the congruence **C** determines a quintic correspondence of precisely the same type as that employed for its generation. The principal points thereof are easily recognised in the Table of Art. 72.

Between the planes of each of the above three pairs, however, the correspondence determined by **C** is quartic, and of the Jonquierian type (Art. 29). The principal triple points thereof are, of course, the vertices of the two cubic congruence-cones; and, of the six principal single points of each plane, two are coincident, with one another and with their associates, at each of the two singular points of **C** which are common to the pair of planes under consideration. Of the three remaining singular points of **C** situated in each of these planes, two are principal single points of the quartic correspondence between them, and the third is the point towards which converge all lines joining the points of the intersection of the planes to their several corresponding points. The latter, together with the point of convergence and the principal points of the plane, are all situated on a conic, seeing that the above intersection must be regarded as a line passing through two

principal single points only; the points respectively coincident with these being, in reality, consecutive points thereto, situated on the principal cubic of the plane to which the intersection is supposed to belong.

Finally, between the planes of each of the remaining six pairs the congruence **C** determines a quartic correspondence of the type considered in Art. 66; but of a special form. Of the two singular points of **C** common to the planes of the pair, that which is the centre of a congruence-pencil is simply a self-corresponding point of this quartic correspondence, and with that which is the vertex of a cubic congruence-cone, one of the principal double points of each plane coincides. The other two principal double points situated in each plane are vertices of quadric congruence-cones in **C**, and their associated principal single points (Art. 66) in the quartic correspondence, are both coincident with the two coincident principal double points. The principal conic corresponding to each of the latter breaks up into the pair of principal lines which there intersect, and themselves correspond to the principal single points coincident with the principal double point in question.

Of the remaining two singular points of **C** situated in each of the two planes, one is the principal single point, of the quartic correspondence, which is associated with one of the two coincident double points, and the other (centre of a congruence-pencil of **C** situated in the plane under consideration) is the point of convergence of all congruence-lines joining points of the intersection of the pair of planes with their correspondents, situated on a conic.

72. Returning once more to the quintic correspondence between α and β by means of which the congruence **C** of Art. 70 was generated, I observe that the latter may be resolved, in three different ways, into a system of quadric reguli. Their respective directrices are the corresponding elements of the following pencils:—

$$\begin{aligned} &(A_3) && \text{and} && (B_3B_2B_2'B_2''),\\ &(A_3A_2A_2'A_2'') && \text{and} && (B_3),\\ &(A_3A_2'A_2''A_1) && \text{and} && (B_3B_2'B_2''B_1). \end{aligned}$$

The reguli conjugate to those included in each of the above three systems form, in the aggregate, another congruence (2, 4) having, with **C**, the same focal surface. In the following Table the latter are denoted by $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, respectively. The rays of any two of the four congruences group themselves into conjugate reguli situated on a system of quadric surfaces enveloping the common focal surface, passing through eight fixed points, and touching two fixed planes. Amongst the quadrics of each of the six systems, above referred to, there are

always two point-and-plane pairs, and two cones. A glance at the columns of the Table appropriated to the two congruences under consideration is sufficient to recognise the points of the former (Art. 68), and on each of the two lines not appropriated by any of their planes will be found, in the last two columns, the vertices of the latter. The planes to which the two last-mentioned lines refer, are those which are touched by every quadric of the system, and the eight points through which all these surfaces pass are indicated on the two lines appropriated to the planes of each of the two point-and-plane pairs.

It should be observed that the two quadric cones common to any two of the four congruences, though they have the same vertices, are different from the two cones common to the remaining two congruences.

The vertices of the two cubic cones of any one of the four congruences will be found in the appropriate column, on the last two lines of the Table. Each of these cones passes twice through the vertex of the other, once through each of the vertices of the six quadric cones of the congruence, and once through each of the centres of three of its pencils—the three, in fact, whose planes pass through the vertex of the cubic cone under consideration.

	$\mathbf{C}$	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$		
α	B_2	A_1	A_3	A_2	A'_2	A''_2
β	A_2	B_3	B_1	B_2	B'_2	B''_2
α_1	A_1	B_2	C	B_3	C'	C''
β_1	B_1	D	A_2	A_3	C'	C''
γ	C	A_3	A_1	D	B'_2	B''_2
δ	D	B_1	B_3	C	A'_2	A''_2
	A_3	A_2	B_2	A_1		
	B_3	C	D	B_1		

73. From the fact that the sextic correspondence by means of which the congruence (2, 5) is to be generated (Art. 64), must have one principal double point and three principal single ones, of each plane, situated on their intersection $\overline{\alpha\beta}$, we at once recognise the correspondence to be the remarkable one whose principal points are not of the same nature in both planes.* In one of these, say α, there are, in fact, three triple, one double, and four single points, whilst in the other, β, one of the principal points is quadruple, four are double, and three single.

* *Bulletin*, Art. 22.

The association of these points is readily effected. To each single point of α corresponds the principal line joining its associated double point in β to the quadruple point; whilst to each single point of the latter plane corresponds the principal line passing through its two non-associated triple points in α.

In conformity with previous practice, therefore, we may denote the several pairs of associated principal points thus:

$$A_1, B_2;\ A_1', B_2';\ A_1'', B_2'';\ A_1''', B_2''';\ A_3', B_1';\ A_3'', B_1'';\ A_3''', B_1''';$$

and the remaining two principal points of the correspondence by A_2 and B_4.

The principal points situated on $\overline{\alpha\beta}$ being A_2, A_1', A_1'', A_1''', B_2, B_1', B_1'', B_1'''—each situated, of course, on its corresponding principal conic or line—it is at once manifest that A_2 and B_2 will be centres of congruence-pencils situated, respectively, in β and α, and that A_1 will be the centre of a third such pencil situated in the plane $(A_1B_2B_4)$, which we will denote by γ. Moreover, B_4 is the vertex of a quartic congruence-cone which traces, on α, the principal curve corresponding to its vertex; viz., a quartic having double points at A_3', A_3'', A_3''', and passing likewise through A_2, A_1, A_1', A_1'', and A_1'''. The points A_3', A_3'', A_3''' are vertices of cubic congruence-cones, each of which traces, on β, the principal curve corresponding to its vertex;—that is to say, a cubic passing twice through B_4, and once through all the other principal points of the plane, the single one excepted which is associated with the vertex. In like manner, B_2', B_2'', B_2''' are vertices of quadric congruence-cones; the principal conic which each traces, on α, passes through the principal single point associated with its vertex, as well as through the double point A_2, and the triple points A_3', A_3'', A_3'''.

Besides the above three quadric cones, however, the congruence **C** now under consideration possesses three others. Their vertices C', C'', C''' are in the plane γ, and they belong to one of the two systems of quadric reguli into which **C** may be resolved (see Art. 75), and whose directrices are corresponding conics of the pencils

$$(A_3'A_3''A_3'''A_1) \text{ and } (B_4B_2'B_2''B_2''').$$

In fact, the generator, passing through A_1, of every such regulus must lie in the plane γ of that point and its corresponding line $\overline{B_2B_4}$, and the generator of it which passes through B_4 must belong to the quartic congruence-cone above described, which has this point for vertex. But the regulus will, obviously, become a cone when these two generators are incident with each other, and this will happen when the one last mentioned coincides with any one of the intersections, exclusive of $\overline{B_4A_1}$, of the above quartic congruence-cone and the plane γ. It can readily be shown that the vertices C_1', C_1'', C_1''' lie, with the principal points A_1, B_2, B_4, on a conic.

74. Six of the thirteen singular points of **C** lie in each of its three singular planes; one, B_2, being common to all three, and three others, A_2, B_4, A_1, to the pairs of planes α, β; β, γ; and γ, α, respectively. Between the planes, γ and α, of the last pair, **C** determines a sextic correspondence precisely like that from which it was generated. Between the planes β and γ, however, **C** determines a quintic correspondence of the type described in Art. 70, but of a very special kind. The principal triple points thereof both coincide, in fact, with B_4; the principal double points are at B_2', B_2'', B_2''' and C', C'', C'''; but the principal single points respectively associated with these are all coincident with the two coincident triple points at B_4. One consequence of this very special disposition of principal points is, that each of the principal cubics of the correspondence breaks up into three right lines, coincident with the principal lines of its plane. The point B_2, centre of a congruence-pencil in α, is now a self-corresponding point, and to $\overline{\beta\gamma}$, regarded as a line passing through the triple point of either plane, corresponds, point by point, a conic which passes through the three double points of the other, as well as through the self-corresponding point B_2, and the triple point at B_4. The latter point being likewise self-correspondent, the congruence-lines joining the several points of $\overline{\beta\gamma}$ with their corresponding ones all converge to a point on the above conic (Art. 51);—in fact, to A_2 in β, and to A_1 in γ.

75. Returning now to the sextic correspondence of Art. 73, I observe that the congruence **C** may be resolved, in two different ways, into a system of quadric reguli. By the one way, the directrices of these reguli are the several pairs of corresponding conics of the pencils

$$(A_3' A_3'' A_3''' A_1) \text{ and } (B_4 B_2' B_2'' B_2'''),$$

already alluded to in the last Article; by the other, they are corresponding elements of the pencils

$$(A_3' A_3'' A_3''' A_2) \text{ and } (B_4).$$

Each system of reguli includes three cones and one pencil-pair. In the first system the cones have their vertices at C', C'', C'''; in the second at B_2', B_2'', B_2'''. The pencil-pair, in the first system, is $(A_2\beta)$, $(B_2\alpha)$, and proceeds from the corresponding conics $(A_3' A_3'' A_3''' A_1 A_2)$ and $(B_4 B_2' B_2'' B_2''' B_2)$. In the second system it is $(A_1\gamma)$, $(\overline{B_2\alpha})$, proceeding from the corresponding elements $(A_3' A_3'' A_3''' A_2 A_1)$ and $\overline{B_4 B_2}$ of the generating pencils.

The reguli conjugate to those included in each of the above systems form, in the aggregate, another congruence (2, 5), having the same focal surface as **C**. These congruences, in the following Table, are denoted by $\mathbf{C}_1$ and $\mathbf{C}_2$. The rays of any two of the three associated congruences group themselves, in fact, into conjugate reguli, situated on a system of quadric surfaces passing, individually, through eight fixed

points, and touching one fixed plane. Each system, moreover, includes one point-and-plane pair, and three quadric cones.

Given any two of the three congruences, the elements of the point-and-plane pair are at once recognised in the Table in the usual manner (Art. 68); the remaining eight points indicated on the two lines appropriated to the planes of this point-and-plane pair, are those through which every quadric of the system of quadrics to which it belongs passes. The singular plane touched by every one of these quadrics, is that to which the remaining line of the Table is appropriated; on it, in the right-hand margin, will be found the vertices of the three quadric cones, whose generators are common rays of the two congruences under consideration.

In this same margin of the Table, and on each of the remaining two lines thereof, are the three singular points which are vertices of cubic cones in one of the two congruences (that indicated by the column which intersects the line in question in B_2), and of quadric cones in the other (Art. 36 **c**). Of the three singular points, lastly, which stand in the margin at the bottom of the Table, each is the vertex of a quartic cone in the congruence indicated by the column in which it stands.

	C	$\mathbf{C}_1$	$\mathbf{C}_2$			
α	B_2	A_2	A_1	A_3'	A_3''	A_3'''
β	A_2	B_2	B_4	B_2'	B_2''	B_2'''
γ	A_1	B_4	B_2	C'	C''	C'''
	B_4	A_1	A_2			

On the Mutual Potential of Two Lines in Space.

By Horace Lamb, M.A.

[*Read June 14th*, 1883.]

By the mutual potential M of two lines C, C' is meant, in Electromagnetism, that part of the energy of the field which is due to the simultaneous presence of two electric currents of unit intensity flowing along these lines in the positive directions, the positive direction along each line being in the first instance chosen arbitrarily.

When both lines are *closed*, the value of M is known to be

$$M = \iint \frac{\cos\epsilon}{r}\, ds\, ds' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where r is the distance between the elements ds, ds' of the two curves, and ϵ denotes the angle between the positive directions of these elements. The question arises—What forms of M are admissible, consistently with the above result, for *unclosed* curves? This question has been discussed by Helmholtz,* who arrives at the result that the most general value of M is of the form

$$M = \iint \left(\frac{\cos \epsilon}{r} + \frac{d^2\Psi}{ds\, ds'} \right) ds\, ds' \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where Ψ is some function of r. The reasoning by which this result is obtained is of a very simple character, but it involves, as Helmholtz has himself subsequently recognised, the assumption that (1) is to be accepted as the complete expression for M when *one* only of the lines C, C' is closed. In the following investigation the formula (2) is shown to follow from Weber's experimental result that two small electric circuits act on one another like two properly chosen magnets.

Let the relative positions of r, ds, ds' be defined by the same notation as in Maxwell's *Electricity*, § 513; viz., θ, θ' are the angles which ds, ds' respectively make with r, and η is the angle between the planes (r, ds) and (r, ds'). Also, if (x, y, z) be the coordinates of ds, (x', y', z') those of ds', we write ξ, η, ζ for $(x'-x)$, $(y'-y)$, $(z'-z)$, respectively. Then ds may be replaced by its components α, β along and perpendicular to r, viz.,

$$\alpha = \cos\theta\, ds, \quad \beta = \sin\theta\, ds;$$

and ds' by three components α', β', γ', of which α' is along r, β' is parallel to β, and γ' is perpendicular to the plane of α, β, viz.,

$$\alpha' = \cos\theta' ds', \quad \beta' = \sin\theta' \cos\eta\, ds', \quad \gamma' = \sin\theta' \sin\eta\, ds'.$$

Hence the most general expression for the mutual potential of the two elements ds, ds' must be of the form

$$(\alpha, \alpha')\, \alpha\alpha' + (\alpha, \beta')\, \alpha\beta' + (\alpha, \gamma')\, \alpha\gamma' + (\beta, \alpha')\, \beta\alpha' + (\beta, \beta')\, \beta\beta' + (\beta, \gamma')\, \beta\gamma' \quad \ldots\ldots\ldots\ldots (3),$$

where the coefficients (α, α'), &c. are functions of r only. It is easily seen from considerations of symmetry that we must have

$$(\alpha, \beta') = 0, \quad (\alpha, \gamma') = 0, \quad (\beta, \alpha') = 0.$$

Let us further assume for the present that $(\beta, \gamma') = 0$. The formula (2) then reduces to

$$(\alpha, \alpha')\, \alpha\alpha' + (\beta, \beta')\, \beta\beta',$$

or, since $\cos\epsilon = -\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\eta$,

to $$(P \cos\theta \cos\theta' + Q \cos\epsilon)\, ds\, ds' \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

* *Crelle*, t. 72, p. 73.

where $P = (\alpha, \alpha') + (\beta, \beta')$, and $Q = (\beta, \beta')$. Transformed to rectangular coordinates, (4) becomes

$$-\frac{P}{r^2}(\xi dx+\eta dy+\zeta dz)(\xi dx'+\eta dy'+\zeta dz')+Q(dx\,dx'+dy\,dy'+dz\,dz') \quad \ldots\ldots(5),$$

i.e., it

$$= X'dx'+Y'dy'+Z'dz' \quad \ldots\ldots(6),$$

where

$$\left.\begin{aligned} X' &= -\frac{P\xi}{r^2}(\xi dx+\eta dy+\zeta dz)+Q dx \\ Y' &= -\frac{P\eta}{r^2}(\xi dx+\eta dy+\zeta dz)+Q dy \\ Z' &= -\frac{P\zeta}{r^2}(\xi dx+\eta dy+\zeta dz)+Q dz \end{aligned}\right\} \quad \ldots\ldots(7).$$

If we integrate (6) round an elementary circuit situate at (x', y', z') whose area is dS', and aspect (λ', μ', ν'), we obtain, on the usual convention as to the relation between that direction of the normal to which λ', μ', ν' refer, and the positive direction round the circuit,

$$\left\{\lambda'\left(\frac{dZ'}{dy'}-\frac{dY'}{dz'}\right)+\mu'\left(\frac{dX'}{dz'}-\frac{dZ'}{dx'}\right)+\nu'\left(\frac{dY'}{dx'}-\frac{dX'}{dy'}\right)\right\} dS' \ldots(8).$$

Now $\dfrac{dZ'}{dy'}-\dfrac{dY'}{dz'}=\dfrac{dZ'}{d\eta}-\dfrac{dY'}{d\zeta}=-\left(\dfrac{1}{r}\dfrac{dQ}{dr}+\dfrac{P}{r^2}\right)(\zeta dy-\eta dz)$, &c. &c.

Hence (8) may be written in the form

$$X dx+Y dy+Z dz \quad \ldots\ldots(9),$$

where

$$\left.\begin{aligned} X &= R(\nu'\eta-\mu'\zeta)\,dS' \\ Y &= R(\lambda'\zeta-\nu'\xi)\,dS' \\ Z &= R(\mu'\xi-\lambda'\eta)\,dS' \end{aligned}\right\} \quad \ldots\ldots(10),$$

provided

$$R = -\left(\frac{1}{r}\frac{dQ}{dr}+\frac{P}{r^2}\right) \quad \ldots\ldots(11).$$

If we next integrate (9) round an elementary circuit of area dS and aspect (λ, μ, ν) situate at (x, y, z), we obtain

$$\left\{\lambda\left(\frac{dZ}{dy}-\frac{dY}{dz}\right)+\mu\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)+\nu\left(\frac{dY}{dx}-\frac{dX}{dy}\right)\right\} dS \quad \ldots(12).$$

Now

$$\frac{dZ}{dy}-\frac{dY}{dz}=\frac{dY}{d\zeta}-\frac{dZ}{d\eta}$$

$$=\left\{\left(r\frac{dR}{dr}+2R\right)\lambda'-\frac{\xi}{r}(\lambda'\xi+\mu'\eta+\nu'\zeta)\right\} dS', \text{ &c., &c.}$$

Hence the most general expression for the mutual potential of two

elementary circuits is

$$\left\{\left(r\frac{dR}{dr}+2R\right)(\lambda\lambda'+\mu\mu'+\nu\nu') - \frac{1}{r}\frac{dR}{dr}(\lambda\xi+\mu\eta+\nu\zeta)(\lambda'\xi+\mu'\eta+\nu'\zeta)\right\} dS\,dS' \quad ...(13).$$

In virtue of Weber's experiments, this must be equal to

$$-\left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\nu\frac{d}{dz}\right)\left(\lambda'\frac{d}{dx'}+\mu'\frac{d}{dy'}+\nu'\frac{d}{dz'}\right)\frac{1}{r}\,.\,dS\,dS',$$

i.e., to

$$-\left\{\frac{1}{r^3}(\lambda\lambda'+\mu\mu'+\nu\nu')-\frac{3}{r^5}(\lambda\xi+\mu\eta+\nu\zeta)(\lambda'\xi+\mu'\eta+\nu'\zeta)\right\} dS\,dS' \quad ...(14).$$

In order that (13) and (14) may be identical, we must have

$$R=\frac{1}{r^3},$$

whence

$$P=-\frac{1}{r}-r\frac{dQ}{dr}.$$

Substituting in (4) or (5), and making use in the latter case of the relations

$$\cos\theta=-\frac{dr}{ds},\quad \cos\theta'=-\frac{dr}{ds'},$$

$$\cos\epsilon=-\frac{dr}{ds}\frac{dr}{ds'}+r\frac{d^2r}{ds\,ds'},$$

we readily find for the mutual potential of two elements ds, ds' the value

$$\left(\frac{\cos\epsilon}{r}+\frac{d^2\Psi}{ds\,ds'}\right)ds\,ds',$$

where

$$\Psi=\int_0^r(1-Qr)\,dr \quad .....(15).$$

This agrees with (2).

It remains to discuss the assumption made above, that the coefficient (β, γ') in (3) is zero. This assumption has been made, almost as a matter of course, in all extant theories of electro-dynamics. It implies that the mutual action between two currents remains the same when the geometrical relations between them undergo the process of "perversion." I proceed to examine what would be the effect of the term $(\beta, \gamma')\beta\gamma'$ in the formula (3), if it existed. If we make $T=(\beta, \gamma')$, this term may be written $T\sin\theta\sin\theta'\sin\eta\,ds\,ds'$. Now

$$r\sin\theta\sin\theta'\sin\eta\,ds\,ds'=\begin{vmatrix}\xi, & \eta, & \zeta\\ dx, & dy, & dz\\ dx', & dy', & dz'\end{vmatrix},$$

each side representing six times the volume of the tetrahedron of

which ds, ds' are opposite edges. Hence the term in question becomes

$$\mathfrak{X}'dx' + \mathfrak{Y}'dy' + \mathfrak{Z}'dz' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16),$$

where

$$\left.\begin{aligned} \mathfrak{X}' &= \frac{T}{r}(\eta dz - \zeta dy) \\ \mathfrak{Y}' &= \frac{T}{r}(\zeta dx - \xi dz) \\ \mathfrak{Z}' &= \frac{T}{r}(\xi dy - \eta dx) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17).$$

Integrating (16) round an elementary circuit of area dS' and aspect (λ', μ', ν'), we obtain

$$\left\{\lambda'\left(\frac{d\mathfrak{Z}'}{dy'} - \frac{d\mathfrak{Y}'}{dz'}\right) + \mu'\left(\frac{d\mathfrak{X}'}{dz'} - \frac{d\mathfrak{Z}'}{dx'}\right) + \nu'\left(\frac{d\mathfrak{Y}'}{dx'} - \frac{d\mathfrak{X}'}{dy'}\right)\right\} dS' \ldots (18),$$

where

$$\frac{d\mathfrak{Z}'}{dy'} - \frac{d\mathfrak{Y}'}{dz'} = -\left(r\frac{d}{dr}\frac{T}{r} + 2\frac{T}{r}\right)dx + \frac{\xi}{r}(\xi dx + \eta dy + \zeta dz)\frac{d}{dr}\frac{T}{r}, \text{ \&c., \&c.}$$

Hence (18) may be written

$$\mathfrak{X}dx + \mathfrak{Y}dy + \mathfrak{Z}dz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

where $\mathfrak{X} = \left\{-\lambda'\left(r\frac{d}{dr}\frac{T}{r} + 2\frac{T}{r}\right) + \xi(\lambda'\xi + \mu'\eta + \nu'\zeta)\frac{1}{r}\frac{d}{dr}\frac{T}{r}\right\} dS'$,

$\mathfrak{Y} = \text{\&c.}, \qquad \mathfrak{Z} = \text{\&c.}$

Integrating (19) round an elementary circuit of area dS and aspect (λ, μ, ν), we get

$$\left\{\lambda\left(\frac{d\mathfrak{Z}}{dy} - \frac{d\mathfrak{Y}}{dz}\right) + \mu\left(\frac{d\mathfrak{X}}{dz} - \frac{d\mathfrak{Z}}{dx}\right) + \nu\left(\frac{d\mathfrak{Y}}{dx} - \frac{d\mathfrak{X}}{dy}\right)\right\} dS \ldots (20),$$

where $\frac{d\mathfrak{Z}}{dy} - \frac{d\mathfrak{Y}}{dz} = \frac{\nu'\eta - \mu'\zeta}{r}\frac{d}{dr}\left(r\frac{d}{dr}\frac{T}{r} + 3\frac{T}{r}\right) dS'$, \&c., \&c.

Hence, finally, the term $(\beta, \gamma')\beta\gamma$ in (3) would give rise in the mutual potential of two elementary circuits to a term of the form

$$\left\{\xi(\mu'\nu - \mu\nu') + \eta(\nu'\lambda - \nu\lambda') + \zeta(\lambda'\mu - \lambda\mu')\right\}\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\frac{T}{r} + \frac{3T}{r}\right) dS\, dS'$$
$$\ldots\ldots\ldots\ldots (21).$$

This cannot vanish for every pair of elementary circuits unless

$$\frac{d}{dr}\left(r\frac{d}{dr}\frac{T}{r} + 3\frac{T}{r}\right) = 0,$$

i.e. unless

$$T = Ar + \frac{B}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22),$$

where A, B are constants. It is obvious that we must have $A = 0$,

otherwise the mutual potential of two elements would increase with the distance between them. If B were not zero, then, although the mutual potential of any two circuits which are not intertwined would have the value (1) which is required by experiment, the same would not be true of linked circuits. We should have to add to (1) the term $4\pi nB$, where n denotes the number of times the two circuits are intertwined. This term would not affect the mechanical (ponderomotive) forces between the two circuits supposed traversed by electric currents, but it would affect the electromotive forces of induction. Thus there would be induction between two linked currents occupying what on the ordinary theory are "conjugate" positions,—*e.g.*, between a circular current, and an infinite rectilinear coaxial one.

The formula (2) relates primarily to linear currents of uniform strength along their course. The corresponding formulæ for any distribution of currents in three dimensions may be derived by superposition. If F, G, H denote the components of the electro-magnetic momentum at (x, y, z), and u', v', w' the components of current at (x', y', z'), we have

$$\left.\begin{aligned} F &= \iiint \frac{u'}{r}\, dx'dy'dz' + \frac{d\chi}{dx} \\ G &= \iiint \frac{v'}{r}\, dx'dy'dz' + \frac{d\chi}{dy} \\ H &= \iiint \frac{w'}{r}\, dx'dy'dz' + \frac{d\chi}{dz} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23),$$

where

$$\chi = \iiint \left(u'\frac{d\Psi}{dx'} + v'\frac{d\Psi}{dy'} + w'\frac{d\Psi}{dz'}\right) dx'dy'dz' \quad \ldots\ldots\ldots\ldots (24),$$

the integrations extending over all space for which u', v', w' differ from zero. On the older theories of electricity this space includes only the *conductors* present in the field; but if, with Maxwell, we recognize the existence of currents of electric displacement in the surrounding dielectric, it will embrace this medium as well. If we adopt Maxwell's further assumption, that the flow of electricity everywhere satisfies essentially the "solenoidal condition," the value (24) of χ is zero, and the formulæ (23) then make

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25).$$

The methods of this paper are, however, not appropriate to this case, and the values of F, G, H are then best obtained in a different manner.

It may, however, be pointed out that, even on Maxwell's theory, the formula (2) will hold good approximately for two currents flowing along two *linear* conductors C, C', provided Ψ has the value $r/2$. If

F, G, H be the components of the electro-magnetic momentum at (x, y, z) due to a current i'' flowing along a wire $A'B'$, we have

$$F = F_1 + F_2,$$

where

$$F_1 = i'' \int \frac{dx'}{r},$$

the integration extending along $A'B'$, and

$$F_2 = \iiint \frac{1}{r} \frac{df'}{dt}\, dx'dy'dz',$$

where f' is the x-component of the polarization at the point (x', y', z') of the dielectric. Now, we have, approximately,*

$$f = \frac{1}{4\pi} \left\{ \frac{e'(x-x_2)}{r_2^3} - \frac{e'(x-x_1)}{r_1^3} \right\},$$

where the suffixes $_1$ and $_2$ refer to the points A', B', respectively, and $\mp e'$ are the amounts of electricity accumulated at these points up to the time t. Hence

$$\nabla^2 F_2 = -4\pi \frac{df}{dt}$$
$$= -\frac{de'}{dt} \left\{ \frac{x-x_2}{r_2^3} - \frac{x-x_1}{r_1^3} \right\},$$

where ∇^2 has its usual meaning. The solution of this is

$$F_2 = \frac{1}{2} \frac{de'}{dt} \left\{ \frac{x-x_2}{r_2} - \frac{x-x_1}{r_1} \right\}$$
$$= \frac{i''}{2} \left(\frac{dr_2}{dx} - \frac{dr_1}{dx} \right) = \frac{i''}{2} \int_{A'}^{B'} \frac{d}{dx} \frac{dr}{ds'}\, ds'.$$

Hence

$$F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds} = i'' \int \left(\frac{\cos\epsilon}{r} + \tfrac{1}{2} \frac{d^2 r}{ds\, ds'} \right) ds'.$$

In conclusion, it may be remarked that the methods of this paper have a much wider application than to the particular problem which suggested them. They are applicable to the investigation of the possible forms of any scalar function of the relative configuration of the two lines which is such that it changes sign when the direction of either line is reversed.

---

* The effect of induction on the polarization currents in the dielectric is here neglected.

On Unicursal Twisted Quartics. Part II.

By R. A. Roberts, M.A.

[*Read June 14th*, 1883.]

In a recent number of the *Proceedings* (Vol. xiv., Nos. 196, 197), I considered some properties of the unicursal twisted quartic. I propose in this paper to make some further investigations with regard to surfaces connected with this curve. In what follows I continue the numbering of the equations from my former paper.

To obtain the equation of the unique quadric which in general contains the curve, we multiply each of the equations (1) by λ and μ, respectively, when we get eight equations from which we can eliminate linearly λ, μ, λ^5, $\lambda^4\mu$, $\lambda^3\mu^2$, $\lambda^2\mu^3$, $\lambda\mu^4$, μ^5. We thus obtain the equation of the quadric in the form

$$\begin{vmatrix} x, & 0, & a_1, & 4b_1, & 6c_1, & 4d_1, & e_1, & 0 \\ 0, & x, & 0, & a_1, & 4b_1, & 6c_1, & 4d_1, & e_1 \\ y, & 0, & a_2, & 4b_2, & 6c_2, & 4d_2, & e_2, & 0 \\ 0, & y, & 0, & a_2, & 4b_2, & 6c_2, & 4d_2, & e_2 \\ z, & 0, & a_3, & 4b_3, & 6c_3, & 4d_3, & e_3, & 0 \\ 0, & z, & 0, & a_3, & 4b_3, & 6c_3, & 4d_3, & e_3 \\ u, & 0, & a_4, & 4b_4, & 6c_4, & 4d_4, & e_4, & 0 \\ 0, & u, & 0, & a_4, & 4b_4, & 6c_4, & 4d_4, & e_4 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots (28).$$

This quadric, which we shall call V, is evidently of the sixth degree in the coefficients of the binary quartics f_1, f_2, f_3, f_4.

It may be observed that the coefficient of u^3 in this equation, viz.,

$$\begin{vmatrix} a_1, & 4b_1, & 6c_1, & 4d_1, & e_1, & 0 \\ 0, & a_1, & 4b_1, & 6c_1, & 4d_1, & e_1 \\ a_2, & 4b_2, & 6c_2, & 4d_2, & e_2, & 0 \\ 0, & a_2, & 4b_2, & 6c_2, & 4d_2, & e_2 \\ a_3, & 4b_3, & 6c_3, & 4d_3, & e_3, & 0 \\ 0, & a_3, & 4b_3, & 6c_3, & 4d_3, & e_3 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29),$$

being equated to zero, will give the condition that the plane unicursal quartic represented by the equations $x = f_1$, $y = f_2$, $z = f_3$ should have a triple point. For the quadric V is the locus of the vertices of those cones standing on the curve which have a triple edge, as the generators of V of one system meet the curve three times (see Salmon's *Surfaces*, Art. 380).

For the canonical form (4), if we put

$$\left.\begin{aligned} a_2a_3a_4(\beta-\gamma)^2(\gamma-\delta)^2(\beta-\delta)^2x_1 &= x\\ a_1a_3a_4(\alpha-\gamma)^2(\gamma-\delta)^2(\alpha-\delta)^2x_2 &= y\\ a_1a_2a_4(\alpha-\beta)^2(\beta-\delta)^2(\alpha-\delta)^2x_3 &= z\\ a_1a_3a_4(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2x_4 &= u \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(30),$$

V becomes $$x^2+y^2+z^2+u^2-2\left(1+\frac{2l^2}{5mn}\right)(yz+xu)$$

$$-2\left(1+\frac{2m^2}{5nl}\right)(zx+yu)-2\left(1+\frac{2n^2}{5lm}\right)(xy+zu)=0 \quad \ldots\ldots(31),$$

where $$l=(\alpha-\delta)(\beta-\gamma), \quad m=(\beta-\delta)(\gamma-\alpha),$$
$$n=(\gamma-\delta)(\alpha-\beta).$$

We now mention the two contravariants obtained by equating to zero the invariants S and T of the biquadratic $\lambda f_1+\mu f_2+\nu f_3+\rho f_4$. These two surfaces, which we shall call σ and τ, respectively, are evidently touched by the osculating planes of the curve, and, taken together, will represent the developable surface formed by the tangents. The quadric σ is of the second degree in the coefficients, and is the simplest contravariant the curve has. Since the discriminant of σ must be an invariant of the curve, and the only invariants of the curve are S and T, which are of the eighth and twelfth degrees, respectively, in the coefficients, it follows that this discriminant must coincide with S. We see thus that, when S vanishes, the osculating planes are touched by a conic; and this conic, as we have seen in § 9, is the locus of points through which three tangent lines to the curve pass.

If we put $$s_{pq}=a_pe_q+a_qe_p-4(b_pd_q+b_qd_p)+6c_pc_q,$$

the equation of σ will evidently be

$$s_{11}\lambda^2+\ldots+2s_{12}\lambda\mu+\ldots=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(32);$$

hence we see that S can be written as a symmetrical determinant as follows:

$$S=\begin{vmatrix} s_{11}, & s_{12}, & s_{13}, & s_{14}\\ s_{12}, & s_{22}, & s_{23}, & s_{24}\\ s_{13}, & s_{23}, & s_{33}, & s_{34}\\ s_{14}, & s_{24}, & s_{34}, & s_{44} \end{vmatrix}.$$

For the canonical form, if we put, as at (30),

$$\left.\begin{aligned} a_1\lambda_1(\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2 &= \lambda\\ a_2\lambda_2(\beta-\alpha)^2(\beta-\gamma)^2(\beta-\delta)^2 &= \mu\\ a_3\lambda_3(\gamma-\alpha)^2(\gamma-\beta)^2(\gamma-\delta)^2 &= \nu\\ a_4\lambda_4(\delta-\alpha)^2(\delta-\beta)^2(\delta-\gamma)^2 &= \rho \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(33),$$

we have $\sigma \equiv l^2(\lambda\rho+\mu\nu)+m^2(\mu\rho+\nu\lambda)+n^2(\nu\rho+\lambda\mu)$ .....(34),

$\tau \equiv \mu\nu\rho+\nu\rho\lambda+\rho\lambda\mu+\lambda\mu\nu$.....(35).

We obtain a contravariant σ' of the sixth degree in the coefficients by substituting differential symbols in the canonizant, and operating on the Hessian of $\lambda f_1+\mu f_2+\nu f_3+\rho f_4$. Since a binary quartic coincides with its Hessian, when it is a perfect square, we see from § 4 that this quadric σ' is touched by all the double tangent planes of the curve.

For the canonical form, σ' is

$$l^3(\lambda\rho+\mu\nu)+m^3(\mu\rho+\nu\lambda)+n^3(\nu\rho+\lambda\mu) \quad \text{.....(36),}$$

multiplied by $a_1a_2a_3a_4$.

If we now form the equation in x, y, z, u coordinates of $\sigma+\lambda\sigma'$, we get an expression of the form $s+\lambda\phi+\lambda^2\phi'+\lambda^3 s'$, where s and s' are the corresponding equations of σ and σ', and ϕ, ϕ' are two new quadrics of the tenth and fourteenth degrees, respectively, in the coefficients.

For the canonical form, these quadrics are evidently linear functions of $x^2+y^2+z^2+u^2$, $yz+xu$, $zx+yu$, $xy+zu$, or of the squares of the planes $x+y+z+u$, $x+y-z-u$, $y+z-x-u$, $z+x-y-u$, and any covariant of theirs will be expressible in the same manner. We have, then, two covariant quadrics of the sixth degree in the coefficients, namely, V (28) and s, one of the tenth degree, and one of the fourteenth degree, namely, the conics ϕ and ϕ'; and all other covariant quadrics would appear to be expressible in terms of these four and the invariants S and T.

It may be observed that s for the canonical form is

$$x^2+y^2+z^2+u^2-2\left(1-\frac{2l^2}{mn}\right)(yz+xu)-2\left(1-\frac{2m^2}{nl}\right)(zx+yu)$$
$$-2\left(1-\frac{2n^2}{lm}\right)(xy+zu)\text{.....(37),}$$

and
$$\phi \equiv l^3\{(m^2+n^2)(m-n)^2+3l^4\}(yz+xu)$$
$$+m^3\{(n^2+l^2)(n-l)^2+3m^4\}(zx+yu)$$
$$+n^3\{(l^2+m^2)(l-m)^2+3n^4\}(xy+zu) \text{.....(38)}$$

multiplied by $a_1a_2a_3a_4$.

For the contravariant quadrics, we have σ and σ' of the second and sixth degrees in the coefficients, respectively, and the reciprocal of σ with regard to V which is of the tenth degree; also the reciprocal of σ with regard to ϕ (38) which is of the fourteenth degree. For the canonical form (33) all these quadrics and their covariants are linear functions of the squares of the points $\alpha+\beta+\gamma+\delta$, $\alpha+\beta-\gamma-\delta$, $\alpha+\gamma-\beta-\delta$, $\beta+\gamma-\alpha-\delta$. Hence it appears that all the contravariant quadrics of the curve can be expressed in terms of S and T and the four quadrics found above.

If we substitute differential symbols $\frac{d}{d\lambda}, \frac{d}{d\mu}, \frac{d}{d\nu}, \frac{d}{d\rho}$ for x, y, z, u in the quadric V and operate on τ (35), we obtain a linear contravariant of the ninth degree in the coefficients, which for the canonical form (33) is

$$\lambda+\mu+\nu+\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

multiplied by $(a_1a_2a_3a_4)^2$. This is the only linear contravariant the curve has; and its square is evidently connected linearly with the four contravariant quadrics.

If we substitute differential symbols $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{du}$ for λ, μ, ν, ρ in (39) and operate on V, we obtain a linear covariant of the fifteenth degree in the coefficients, which for the canonical form (30) is

$$(a_1a_2a_3a_4)^3\,(x+y+z+u) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40).$$

This is the only linear covariant the curve has. Its square is connected linearly with the four covariant quadrics V, s, ϕ, and ϕ'.

If we form the Jacobian of the four covariant quadrics V, s, ϕ, and ϕ', we get for the canonical form (30),

$$J \equiv (a_1a_2a_3a_4)^6\,(x+y+z+u)(x+y-z-u)(z+x-y-u)(y+z-x-u),$$

which is of the thirty-sixth degree in the coefficients. But this is divisible by the linear covariant (40); hence we see that

$$(a_1a_2a_3a_4)^3\,(x+y+z+u)(x^2+y^2+z^2+u^2-2xy-2yz \\ -2zx-2ux-2uy-2uz) \\ +8\,(a_1a_2a_3a_4)^3\,(xyz+uyz+uzx+uxy) \quad \ldots\ldots\ldots\ldots\ldots (41)$$

is a covariant of the twenty-first degree; but, from (31) and (37), we have $5V+s = 6\,(x^2+y^2+z^2+u^2-2xy-2yz-2zx-2ux-2uy-2uz)$; thus we see that the first member of (41) is expressible in terms of covariants already found. Hence we may take

$$(a_1a_2a_3a_4)^3\,(xyz+uyz+uzx+uxy) \quad \ldots\ldots\ldots\ldots\ldots\ldots (42)$$

as an independent covariant cubic of the twenty-first degree in the coefficients.

We now proceed to find the equation of the product of the stationary planes. Let $\alpha x+\beta y+\gamma z+\delta u = 0$ be a stationary plane, then we have $\alpha f_1+\beta f_2+\gamma f_3+\delta f_4 = k\,(\lambda\mu'-\lambda'\mu)^4$, from which, by equating the coefficients of λ^4, $\lambda^3\mu$, &c., we get

$$\begin{aligned}
a_1\alpha+a_2\beta+a_3\gamma+a_4\delta &= \quad k\mu'^4,\\
b_1\alpha+b_2\beta+b_3\gamma+b_4\delta &= -k\mu'^3\lambda',\\
c_1\alpha+c_2\beta+c_3\gamma+c_4\delta &= \quad k\mu'^2\lambda'^2,\\
d_1\alpha+d_2\beta+d_3\gamma+d_4\delta &= -k\mu'\lambda'^3,\\
e_1\alpha+e_2\beta+e_3\gamma+e_4\delta &= \quad k\lambda'^4;
\end{aligned}$$

hence, eliminating λ' and μ', we can write down the six equations,

$$(b_1^2 - a_1 c_1)\,\alpha^2 + \ldots + (2b_1 b_2 - a_1 c_2 - a_2 c_1)\,\alpha\beta + \ldots = 0,$$
$$(c_1^2 - b_1 d_1)\,\alpha^2 + \ldots + (2c_1 c_2 - b_1 d_2 - b_2 d_1)\,\alpha\beta + \ldots = 0,$$
$$(d_1^2 - e_1 c_1)\,\alpha^2 + \ldots + (2d_1 d_2 - e_1 c_2 - e_2 c_1)\,\alpha\beta + \ldots = 0,$$
$$(b_1 c_1 - a_1 d_1)\,\alpha^2 + \ldots + (b_1 c_2 + b_2 c_1 - a_1 d_2 - a_2 d_1)\,\alpha\beta + \ldots = 0,$$
$$(c_1 d_1 - e_1 b_1)\,\alpha^2 + \ldots + (c_1 d_2 + c_2 d_1 - e_1 b_2 - e_2 b_1)\,\alpha\beta + \ldots = 0,$$
$$(a_1 c_1 - b_1 d_1)\,\alpha^2 + \ldots + (a_1 e_2 + a_2 e_1 - b_1 d_2 - b_2 d_1)\,\alpha\beta + \ldots = 0.$$

From these equations, together with the four obtained by multiplying $ax + \beta y + \gamma z + \delta u = 0$ by $\alpha, \beta, \gamma, \delta$, respectively, we can eliminate α^2, β^2, $\alpha\beta$, &c., linearly, when we obtain the result in the form of a determinant. This covariant, which we shall call Φ, is evidently of the twelfth degree in the coefficients. For the canonical form (30), $\Phi \equiv xyzu$.

It may be observed that the coefficient of u^4 in the determinant found above, being equated to zero, will give the condition that the plane curve determined by $\frac{x}{f_1} = \frac{y}{f_2} = \frac{z}{f_3}$ should have a point of undulation.

If we substitute differential symbols in σ and operate on Φ, we get a covariant quadric of the fourteenth degree in the coefficients which might conveniently replace the quadric ϕ'.

For the canonical form, this quadric is

$$l^2(yz + xu) + m^2(zx + yu) + n^2(xy + zu) \quad \ldots\ldots\ldots\ldots (43).$$

If we substitute differential symbols in the linear contravariant (39), and operate on Φ, we obtain the cubic covariant (42) directly.

We have already obtained a cubic contravariant τ (35) of the third degree in the coefficients. The equation of this surface in x, y, z, u coordinates is easily proved to be

$$x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} + u^{\frac{1}{3}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (44),$$

or $$(x^2 + y^2 + z^2 + u^2 - 2xy - 2yz - 2zx - 2ux - 2uy - 2uz)^2 - 64xyzu = 0 \ldots\ldots\ldots\ldots\ldots (45).$$

This surface is, then, a Steiner's quartic, and from (45) we see that it can be expressed in terms of Φ and the quadrics V and s. From the canonical forms of the equation of the curve (4) and (30), we see that every point of the curve satisfies (44). But, from the mode of generation of the surface, we see that it is touched by the osculating planes of the curve; the osculating planes to the curve are, therefore, tangent planes to the surface at every point of its length. Hence the given curve is an asymptotic curve of the surface; that is, touches an inflexional tangent of the surface at every point of its length.

It can be readily deduced from the above that the Steiner's quartic (44), being expressed by means of the equations $x = (1+pq)^4$, $y = (1-pq)^4$, $z = (p+q)^4$, $u = (p-q)^4$, the curves $p =$ a constant, $q =$ a constant, will represent the two systems of asymptotic lines on the surface.

To find an independent contravariant of the fourth order, we substitute the first differential coefficients of τ (35) in the quadric $5V+s$, namely, $x^2+y^2+z^2+u^2-2xy-2yz-2zx-2ux-2uy-2uz$, when we get a result proportional to

$$\Sigma\lambda^2\mu\nu+3\lambda\mu\nu\rho-(\lambda+\mu+\nu+\rho)\,\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots(46).$$

This expression is of the twelfth degree in the coefficients, and $\Sigma\lambda^2\mu\nu$ is the Hessian of τ, and is, therefore, also of the twelfth degree. Now $(\lambda+\mu+\nu+\rho)\,\tau$ is the product of τ by the linear contravariant (39); it is, therefore, of the twelfth degree. Hence, from (46), we see that $\lambda\mu\nu\rho$, or the product of the equations of the vertices of the canonical tetrahedron, is a contravariant of the twelfth degree in the coefficients.

To find the equation of the developable surface formed by the tangent lines to the curve, we take the envelope of the osculating plane, whose equation for the general form is

$$\begin{vmatrix} x, & y, & z, & u \\ x_{\lambda\lambda}, & y_{\lambda\lambda}, & z_{\lambda\lambda}, & u_{\lambda\lambda} \\ x_{\lambda\mu}, & y_{\lambda\mu}, & z_{\lambda\mu}, & u_{\lambda\mu} \\ x_{\mu\mu}, & y_{\mu\mu}, & z_{\mu\mu}, & u_{\mu\mu} \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(47),$$

where the suffixes refer to differentiation. This equation is of the sixth degree in $\lambda : \mu$, and the calculation of its discriminant would be a work of great length. For the canonical form, however, we can obtain the result by means of the method employed by Dr. Salmon for finding the equation of the centro-surface of a quadric. For the equations (4), (47) becomes

$$\frac{x_1}{A(\vartheta-\alpha)^2}+\frac{x_2}{B(\vartheta-\beta)^2}+\frac{x_3}{C(\vartheta-\gamma)^2}+\frac{x_4}{D(\vartheta-\delta)^2}=0 \quad \ldots\ldots(48),$$

where $A = a_1(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)$, $B = a_2(\beta-\alpha)(\beta-\gamma)(\beta-\delta)$,
$C = a_3(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)$, $D = a_4(\delta-\alpha)(\delta-\beta)(\delta-\gamma)$,

and $$\frac{\lambda}{\mu} = \theta.$$

Putting, then,

$$\frac{x_1}{A(\vartheta-\alpha)}+\frac{x_2}{B(\vartheta-\beta)}+\frac{x_3}{C(\vartheta-\gamma)}+\frac{x_4}{D(\vartheta-\delta)}-k=\phi,$$

we must have, from (48),

$$\frac{d\phi}{d\vartheta}=0, \quad \frac{d^2\phi}{d\vartheta^2}=0;$$

ϕ, therefore, must have three roots equal. But, ϕ being a biquadratic, we have the invariants $S = 0$, $T = 0$, which are of the second and third degrees, respectively, in k. We find, then, the required result by eliminating k between these equations.

The following presents were received in the Recess:—

"Proceedings of the Royal Society," Vol. xxxv., Nos. 225 and 226.

"Proceedings of the Society for Psychical Research," Vol. i., Parts 2 and 3; 8vo, London, 1883.

"Proceedings of the Physical Society of London," Vol. v., Part 4.

"Journal of the Institute of Actuaries," Vol. xxiv., Part 1.

"Mathematical Questions, with their Solutions, from the 'Educational Times,'" Vol. xxxix.

"Educational Times," for July, August, September, and October.

"Royal Irish Academy, Proceedings (Science)," Vol. iii., Ser. 2, Nos. 9 and 10; "Polite Literature and Antiquities," Vol. ii., Ser. 2, No. 4.

"Royal Irish Academy, Transactions (Science)," Vol. xxviii., Nos. xi., xii., and xiii.; "Polite Literature and Antiquities," Vol. xxvii., Part 5.

"The Parallax of α Lyrae and 61 Cygni," by Asaph Hall; 4to, Washington, 1882.

"The Mathematical Magazine," Vol. i., No. 7; July, 1883.

"Johns Hopkins University Circulars," Vol. ii., Nos. 24 and 25.

"American Journal of Mathematics," Vol. v., No. 2.

"Telegraphic Determination of Longitude, in the East Indies, China, and Japan"; 4to, Washington, 1883.

"Studies in Logic," by Members of the Johns Hopkins University; 8vo, Boston, 1883.

"Annuaire de l'Académie Royale, &c., de Belgique," 48th and 49th years.

"Bulletins de l'Académie Royale, &c., de Belgique," 3^{me} Série, Tom. 1 to 5, 50^{me} année.

"Memorie della Regia Accademia di Scienze, Lettere ed Arti, Modena," Serie 2, Vol. i., 1883.

"Atti della R. Accademia dei Lincei—Transunti," Serie 3, Vol. vii., Fasc. 11, 12, 13, and 14.

"Atti della R. Accademia dei Lincei—Memorie," Serie iii., Vols. 11, 12, and 13.

"Beiblätter zu den Annalen der Physik und Chemie," Band vii., Stücke 6, 7, 8, 9.

"Jahrbuch über die Fortschritte der Mathematik," Band xiii., Heft 1.

"R. Istituto Lombardo, &c.,—Rendiconti," Serie ii., Vol. xiv.

"R. Istituto Lombardo—Memorie," Vol. xiv., della Serie iii., Fasc. iii.

"Sitzungsberichte der K. Preussischen Akad. der Wissenschaften zu Berlin," Nos. xxii. to xxxvii.

"Bulletin des Sciences Mathématiques et Astronomiques," 2^{me} Série, Tom. viii., April to July, 1883.

"Journal für die reine und angewandte Mathematik," Bd. xciv., Heft 4, Bd. xcv., Heft 1.

"Bulletin Astronomique et Météorologique de Rio de Janeiro," No. 4, (No. 5 missing,) No. 6 (two copies), No. 7.

"Mémoire sur les Faisceaux de Formes Binaires ayant une même Jacobienne," par M. Cyparissos Stephanos; 4to, Paris, 1883.

"Bulletin de la Société Mathématique de France," Tome xi., No. 2.

The following is a list of the Persons, Societies, and Public Institutions to whom the Society forwards copies of its *Proceedings*:—

The Royal Society.
The Royal Society of Edinburgh.
The Royal Irish Academy, Dublin.
The Library of Trinity College, Dublin.
The Cambridge Philosophical Society.
The Philosophical Society of Manchester.
The Institute of Actuaries.
The Library, University College, London.
The British Museum.
The University Library, Cambridge.
The Bodleian Library, Oxford.
The Smithsonian Institute, Washington.
The United States Naval Observatory, Washington.
The Connecticut Academy, Newhaven, U.S.A.
L'Académie des Sciences, Paris.
La Société Mathématique, Paris.
La Société Philomathique, Paris.
M. le Général commandant l'École Polytechnique, Paris.
La Société des Sciences Physiques et Naturelles de Bordeaux.
L'Académie Royale des Sciences, des Lettres, et des beaux Arts de Belgique, Bruxelles.
Le Bureau Scientifique Central Néerlandais, Haarlem.
The Editor of the "Annali di Matematica," Milano.
The Editor of "Nature."
The Editor of "The American Journal of Mathematics," Baltimore.
The Editor of the "Beiblätter zu den Annalen der Physik und Chemie," Leipzig.
The Editor of the "Jahrbuch über die Fortschritte der Mathematik," Berlin.
The Editor of *Crelle*, "Journal für die reine und angewandte Mathematik," Berlin.
The Editor of "Jornal de Sciencias Mathematicas," Coimbra.
The Editor of the "Acta Mathematica," Berlin.
The Editor of the "Bulletin des Sciences Mathématiques et Astronomiques," Paris.
Il Istituto Lombardo, Milano.
R. Accademia dei Lincei, Roma.
R. Accademia di Scienze, Lettere ed Arti, Modena.
Die Naturforschende Gesellschaft, Zürich.
The Library of the University of Erlangen.
Die Akademie der Wissenschaften, Berlin.
Die Königliche Gesellschaft, Göttingen.
Die Königliche Sächsische Gesellschaft, Leipzig.
R. Accademia delle Scienze di Napoli.
The Library of the University, Leiden.
R. Istituto Veneto di Scienze, Lettere ed Arti.

APPENDIX.

In the *Bulletin des Sciences Mathématiques et Astronomiques* (2me Série, Tome vii., Mai, 1883, pp. 145—160) is printed a very full "Liste des Travaux sur les Systèmes Articulés," from 1796—1882, by M. V. Liguine, which contains 151 entries.

A reference to Mr. Kempe's "Map-colouring" (*Proceedings*, Vol. x., pp. 229—231) is made by Prof. Tait in his Introductory Address to the Edinburgh Mathematical Society, Nov. 9th, 1883. The Address, under the heading "On Listing's Topologie," will be found in the *Philosophical Magazine*, Vol. xvii., Jan., 1884 (No. 103, pp. 30—46).

Mr. Tucker's communication on "Two Concentric Circles" (p. 202) is published in the *Quarterly Journal of Mathematics* (Vol. xix., No. 76), under the heading "The Triplicate-Ratio Circle." A résumé of the principal properties arrived at in the article, with some additional (hitherto unpublished) ones, and a few references to other papers are given in the following paragraphs.

The circle is obtained in the following manner. Through a point (P), in the plane of the triangle ABC, straight lines DPE', EPF', FPD' are drawn parallel to the sides AB, BC, CA respectively, and so that the points D, D', E, E', F, F' are concyclic.* The circle obtains its name from the fact that

$$DD' : EE' : FF' = a^3 : b^3 : c^3, \ DD' \text{ being } = a^3/\kappa \quad \ldots\ldots(1).\dagger$$

The triangles DEF, $D'E'F'$, ABC are similar to one another, and

$$DE = a\mu = D'F' \quad \ldots\ldots(2).$$

Also
$$DF' = abc/\kappa = ED' = FE' \quad \ldots\ldots(3).$$

If the angles equal to BFD be denoted by ω, we have

$$\cot\omega = \cot A + \cot B + \cot C \quad \ldots\ldots(4);$$

and if APK, BPL, CPM meet the sides in K, L, M respectively, and the $\angle BAK = \theta_1$, $\angle CAK = \theta_1'$, then

$$\cot\theta_1 + \cot\theta_2 + \cot\theta_3 = 3\cot\omega = \cot\theta_1' + \cot\theta_2' + \cot\theta_3' \ldots\ldots(5).$$

---

* In the *Reprint from the Educational Times* (Vol. ix., p. 79, 1868), Mr. Samuel Roberts has given the equation of the *ellipse* passing through the above six points when P is unrestricted, but the *circle* does not appear to have been discussed in any English Journal.

† In this notice $\kappa \equiv a^2 + b^2 + c^2$, $\lambda^2 \equiv a^2b^2 + b^2c^2 + c^2a^2$.

The equation of the circle takes the following forms (in trilinear coordinates):

$$abc\,(\alpha^2+\beta^2+\gamma^2) = \frac{abc}{\kappa}\,(a\alpha+b\beta+c\gamma)^2+a^3\beta\gamma+b^3\gamma\alpha+c^3\alpha\beta \ldots\ldots(6),$$

and $(b^2+c^2)\,\alpha^2+\ldots+\ldots=\{(a^2+b^2)(a^2+c^2)+b^2c^2\}\,\beta\gamma/bc+\ldots+\ldots$ ...(7).

Now, if through the points A, B, C we draw lines parallel to the sides of the triangles DEF, $D'E'F'$, these lines intersect in two points O, O', and intersect again in three points A_1, B_1, C_1. The points O, O' have been called *Brocard* points, and the angles BAO, CAO', &c. *Brocard's* angles.

The properties of these points are fully discussed in the "Etude d'un nouveau cercle du plan du triangle," par M. H. Brocard (*Association Française pour l'Avancement des Sciences—Congrès d'Alger*, 1881). The Brocard's circle passes through the above five points (O, O'; A_1, B_1, C_1), and also through P and the circumcentre (H) of ABC. Its equation takes the forms

$$abc\,(\alpha^2+\beta^2+\gamma^2) = a^3\beta\gamma+b^3\gamma\alpha+c^3\alpha\beta \ldots\ldots\ldots\ldots(8),$$

and $$\kappa\,(a\beta\gamma+\ldots+\ldots) = (bc\alpha+\ldots+\ldots)(a\alpha+\ldots+\ldots) \ldots\ldots(9).$$

From (6) and (8) we see that the two circles are *concentric*, the centre bisecting the line HP in O_1. The point P is readily seen to be a Brocard point for each of the triangles DEF, $D'E'F$, its character being further specialized by Miss C. A. Scott as the *second* B. point for DEF and the *first* B. point for $D'E'F'$ .....(10).*

This point P is also the radical centre of the circles about $BCE'F$, $AED'B$, $ACDF'$ .....(11).

The "Triplicate-Ratio" circle is a particular case of a group of circles, any one of which cuts the sides of a triangle, so that DE', EF', FD' are parallel to AB, BC, CA respectively, and at distances from those sides in the proportion of the same sides. Hence $BD = tac^2$, $CD' = tab^2$, &c. (see "A Group of Circles," *Quarterly Journal of Mathematics*, Vol. xx., No. 77).

In this case (3) becomes

$$DF' = tabc = ED' = FE';$$

$$(2)\quad D'F'^2 = a^2\,(t^2\lambda^2-t\kappa+1);$$

and (6) becomes

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta = t\,(a\alpha+\ldots+\ldots)\,\{abc\,(1-ta^2)+\ldots+\ldots\}.$$

---

* O (got by cyclic order $OAB = OBC = OCA$) Miss Scott calls the first B. point, and O' the second B. point. We might also name them cyclic and contra-cyclic Brocard points. The result given above was communicated, with geometrical proofs of many of the properties of the "T. R." circle, to the *Educational Times* for November, 1883.

If the lines DE', &c. intersect in P_1, P_2, P_3, the

$$\Delta P_1P_2P_3 = \Delta ABC\,(1-ta)^2,$$

and the equation to the circle $P_1P_2P_3$ is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (a\alpha + b\beta + c\gamma)\,\{ta\,(b\beta + c\gamma) - \alpha\,(1-ta^2)\} = 0,$$

or, symmetrically,

$$3(a\beta\gamma + \ldots + \ldots) + (a\alpha + \ldots + \ldots)\{t\,(a+b+c)(a\alpha + b\beta + c\gamma) - (\alpha + \beta + \gamma)\}.$$

The centre of any circle of this group is the mid-point of the line joining the centre of $P_1P_2P_3$ to the circumcentre (H) of ABC ...(12).

When $t\kappa = 1$, the triangle $P_1P_2P_3$ becomes the point P, and the group-circle the "T. R." circle.

In the general case, we have

$$t\kappa \cot BFD = \pm\,(2-t\kappa)\cot\omega\,;$$

hence, when $t\kappa = 1$, BFD becomes ω.

The angle $DEO = \omega$, therefore O is a first Brocard point of the triangle DEF, and similarly O' is a second Brocard point of the triangle $D'E'F'$, whence we can readily prove that OO_1P, $O'O_1P$ are the Brocard circles respectively of the triangles DEF, $D'E'F'$ .....(13).

It also easily follows, by drawing lines $B'AA'$, $C'BB'$, $A'CC'$ externally at angles inclined at an $\angle\omega$ to the sides of ABC, that we can escribe a triangle $A'B'C'$ similar to ABC, and having O for a first Brocard point, and also, in like manner, a second triangle $A''B''C''$ of which O' is the second Brocard point. Thus we see that O, O' are Brocard points for any number of triangles escribed in the above manner, and P alternately "symmedian" point and second or first Brocard point ...(14);

[*i.e.*, when we have a single escribed triangle, the points will be as in Fig. 19, *Quar. Jour.*, *l.c.*; and, when we have two escribed triangles, the points will be as in (10) above.]

In the article (*Quar. Jour.*, No. 76) it is shown that O_1 is the centre of the Brocard circle, of the "T. R." circle, of the circle through the feet of perpendiculars from O, O' on the sides of ABC (*Lady's and Gentleman's Diary*, 1859), and of the in-circle of the triangle formed by producing the lines DF', FE', ED'.....(15).

We may mention here that it is also the centre of three other circles related to Fig. 19 (*Quar. Jour.*, Vol. xix.). Let

$$\left.\begin{array}{l} DE,\ F'D' \text{ cut in } p \\ EF,\ D'E' \quad ,,\ q \\ FD,\ E'F' \quad ,,\ r \end{array}\right\} \qquad \left.\begin{array}{l} EF,\ E'F' \text{ in } p' \\ FD,\ F'D' \quad ,,\ q' \\ DE,\ D'E' \quad ,,\ r' \end{array}\right\},$$

then the sides of pqr are parallel to those of ABC, and

$$qr = abc \sin A / \kappa \sin 2\omega = a\lambda^2/\kappa^2 = a\mu^2;*$$

hence the circles about $qE'Fr$, $rF'Dp$, $pD'Eq$, are equal (radius $= abc/2\kappa \sin 2\omega$) and cointersect in O_1 .....(16).

If O_a, O_b, O_c are the centres of these circles respectively, then sides of $O_aO_bO_c$ are parallel to sides of ABC, and O_1 is its circumcentre...(17).

It is obviously also the centre of a circle touching the circles (O_a), (O_b), (O_c) externally, and also of the in-circle of the triangle formed by drawing tangents to these circles .....(18).

Further, O_1 is the orthocentre of pqr.....(19).

The circles round AFE, BFD, CDE cointersect in O, and the circles round $AF'E'$, $BF'D'$, $CD'E'$ cointersect in O'.....(21),

and we can readily show that OD, OE, OF are equally inclined to sides of ABC, as are $O'D'$, $O'E'$, $O'F'$, and the lines joining the centres of the two sets of circles form two triangles which are similar to ABC and equal to one another (the sides being $1/2 \sin 2\omega$, those of the primitive triangle) .....(22).

We have the coordinates of

$$A_1 \text{ (on } EF') = abc \sin A/\kappa, \quad c^3 \sin A/\kappa, \quad b^3 \sin A/\kappa,$$

whence, with analogous values for B_1, C_1, we get the a coordinate of centroid of $A_1B_1C_1 = 2\Delta/3a$, *i.e.*, the centroid in question is coincident with the centroid of ABC (a result which is § 15 of Brocard's *Etude*). Also $A_1F' = ab^2/\kappa = CD'$; hence, if through D' we draw a parallel to AC, the intersection with EF' gives A_1, and similarly for the other points, or we may get A_1 by dropping a perpendicular from H on EF'.

Dr. Casey, who proposes to add some of the above propositions to his forthcoming edition of the *Sequel to Euclid*, sends the following additional properties :—O is the centre of similitude of the triangles ABC, DEF; O' of the triangles ABC, $D'E'F'$; P is the centre of perspective of the triangle ABC and of the triangle formed by producing the three chords DF', FE', ED'; the polar of P with respect to the

---

* The segments made by $D'F$, $E'D$ on qr (reckoning from r) are

$$a^3c^2/\kappa^2, \quad ab^2c^2/\kappa^2, \quad a^3b^2/\kappa^2;$$

we have also pPp', qPq', rPr' concurrent; hence the triangles pqr, $p'q'r'$ are in perspective ..... (20).

"T. R." circle is the radical axis of the circum-circle and the "T. R." circle; the polar of P with respect to the circum-circle is the radical axis of the Brocard circle and the circum-circle. These properties, however, are, we think, contained in some one or more of the papers referred to below.

The point P is called by Continental mathematicians (who employ the letter K to denote it) *point de Grebe*;* as it is also the intersection of what M. d'Ocagne (*Nouvelles Annales*, t. ii., p. 450, 1883) calls *les Symédianes*, we have called it the "Symmedian" point of the triangle. Dr. Casey objects to the term as not being very suitable, and suggests for it "Centre of Symmetry" of the triangle, "because 1° it is the mean centre of its own perpendiculars, 2° these perpendiculars are proportional to the sides, 3° the six triangles which have P as their common vertex, and whose bases are the sides of the hexagon $DD'EE'FF'$, are similar to one another and to the triangle ABC."

The peculiar property of the Brocard angle (ω) occurs for the first time, so far as we know, in English books, in the Question 1933 (*Lady's, &c. Diary*, 1859, *l.c.*), though it may have been given elsewhere.

Professor W. Führmann, of Königsberg, writes that J. H. Van Swinden, in his "Grundbeginsel der Meetkunde," Gröningen, 1833 (translated into German by C. F. A. Jacobi, Jena, 1834), mentions the points (O, O') and gives the relations for $\cot\omega$ and $\operatorname{cosec}^2\omega$. [M. Brocard attributes the first announcement of $\cot\omega =$ &c. to the *Problèmes de Trigonométrie* of Léonce Clarke, 1849.]

The earliest writer who gives properties of the circle here designated the "T. R." circle appears to be M. E. Lemoine, in two Notes read before the *Association Française pour l'Avancement des Sciences*, in 1873 at Lyons, in 1874 at Lille.† The only account we have seen of his results is given in the *Nouvelles Annales* (Jan. 1884), by M. d'Ocagne, who records properties of the lines he calls Symmedians, and appends a few of his novel results.

Professor J. Neuberg is the next writer on our list. His contributions are to the *Nouvelle Correspondance Mathématique* (1879), and to *Mathesis* (Oct., Nov., Dec., 1881). It is this latter paper ("Sur le Centre des Médianes antiparallèles") which the writer has kindly sent us. It is a very elegant pamphlet, and contains many results obtained by M. Brocard. M. Neuberg has also, in a letter, communicated to us some results almost identical with those we had obtained

---

* Archives de Grunert, t. viii.

† Also in the *Nouvelles Annales de Gerono* (1873). This writer calls the lines AP, BP, CP *les médianes antiparallèles* of the triangle ABC, and the point their *centre*: in this nomenclature he is followed by MM. Neuberg, Brocard, and others. We learn that M. Lemoine writes on the "T. R." circle in *Mathesis* (1881, p. 189).

in the opening part of the paper "A Group of Circles," with some others of interest. Professor Neuberg writes:—"Je prépare en ce moment un grand travail sur le triangle, dans lequel je ferai entrer les propriétés découvertes dans les derniers temps par MM. Brocard, Lemoine, d'Ocagne, &c."* M. Brocard's first contributions appeared, we believe, in the *Nouvelle Correspondance Mathématique* (t. iii., iv., v., and vi.), but his principal work on this subject is the *Etude* cited above, the substance of which was communicated on April 16th, 1881. In addition to these results, the writer has with great courtesy forwarded lithographed copies of his *Nouvelles Propriétés du Triangle*, of his *Démonstration de Propriétés du Triangle*, and numerous notes bearing more especially on the "T. R." circle. Dr. Kiehl, of Bromberg, has sent a paper *Zur Theorie der Transversalen* (which precedes a *Programm*, read apparently on April 8, 1881). In this are some results bearing upon our subject; more especially we find that the property established in the article on the "T. R." circle is true also for the generalized system considered in "A Group of Circles," *i.e.*, a "group-circle" is concentric with the corresponding circle touching the general lines DF', FE', ED'. M. Brocard gives the following list of references: MM. Morel (*Journal de Bourget*, 1883), Lemoine (*id.*, *passim*), Stoll, Kiehl, Führmann, &c. (*Journal de Hoffmann*), and the "Zeitschrift" of Herr Lieber.

The new properties (on the authority of Professor Neuberg) seem to be those marked (4), *i.e.* new for the *inscribed* triangles, (5), (10), (11). We have not come across any of the following results: (12), (13), (14), (16) ... (22).

Our death-roll this Session contains the names of five members, viz., Messrs. Spottiswoode, H. J. S. Smith, C. J. Monro, C. R. Rippin, and E. B. Seitz.

Of Dr. Spottiswoode, P.R.S., no lengthy notice is required in this place. He was elected an ordinary member, June 19th, 1865, and served for some years on the Council, until the press of other business compelled him to retire from the ranks of the Governing Body. He filled the office of President (being our fifth President) in 1870-1, 1871-2; and that of Vice-President in 1866-7, 1867-8, 1869-70, 1872-3, 1873-4, 1875-6, 1876-7. Not to refer to the several obituary notices which appeared in the papers at the time of his death, June 27th, 1883, we may direct the attention of members to the sketch of his Life and Writings, published in *Nature*, April 26th, 1883. To the list of his scientific papers there given by us, we now add

* He will be happy to receive accounts of any contributions on the subject to English or other journals. The paper is to appear shortly in successive numbers of *Mathesis*.

the title of his last short communication made to the R. Accademia dei Lincei, on May 5th, 1883, "Sur les Invariants et les Covariants d'une Fonction transformée par une substitution quadratique."

Professor Cremona, in his "Cenno Necrologico," calls him "caro e indimenticabile Spottiswoode," and well remarks: "A Spottiswoode furono concessi i più grandi onori che un uomo di scienza in Inghilterra possa ottenere: la presidenza della Società Reale negli ultimi anni di sua vita, e, dopo morte, la tomba nell' Abbazia di Westminster, dove già da circa tre secoli riposa un altro Spottiswoode, arcivescovo di St. Andrews." We believe that it is proposed to reprint in a connected form many, if not all, of his Mathematical and Scientific Papers.

"On Friday, the 9th inst." (Feb. 1883), "we lost one of our most gifted men. By the death of Professor Henry Smith, there has dropped out from our roll-call a name which was already known among a wide circle of friends and admirers, but which would assuredly have been more widely known and more fully recognised if he had remained longer in our ranks." Thus writes Mr. Spottiswoode (*Nature*, Feb. 22, 1883) of one who was one of his intimate friends, and whom he was so soon to follow to that bourne, "unde negant redire quenquam."

Professor Smith joined us on Oct. 16th, 1865, and was always one of our most appreciated readers of papers. The charm of his manner will long abide in the memories of those who heard him; almost the last, if not his last, communication to any Society, was that which he read at our December (14th, 1882) meeting, entitled "On the absolutely Least Periods of the Elliptic Functions." He never appeared to greater advantage than on that occasion, and his words still seem to ring in our ears—

"Oh, for the touch of a vanished hand,
For the sound of a voice that is still!"

Always ready to lay himself out for the benefit of this Society, he not only cheerfully came in reply to any urgent appeal for a paper, but was a most frequent referee, never declining this responsible office on the plea of pressure of other urgent business, which he might so easily and properly have put forward. He served as President (our seventh) in the Sessions 1874-5, 1875-6, and as Vice-President in 1867-8, 1868-9, 1870-1, 1871-2, 1876-7, 1877-8, and in some other Sessions as an ordinary Member of the Council.

In addition to a just estimate of Prof. Smith's character, career, and work in the *Times*, reference may be made here to appreciative notices in the *Spectator*, *Athenæum*, and *Academy*, of Feb. 17th, 1882. The fullest account, and the one most likely to commend itself to our

members, is that furnished to the *Fortnightly Review* (No. cxcvii., pp. 653—666, May 1, 1883), by Mr. J. W. L. Glaisher, F.R.S., who s engaged upon an edition of the mathematical papers and memoirs. These, we understand, will include a considerable quantity of hitherto unpublished matter, and will be preceded by a Biographical Introduction by Miss Smith.

Mr. C. J. Monro was born August 24th, 1833, and died at Hadley, Barnet, Nov. 25th, 1882. He graduated 38th Wrangler in 1855, and was elected a Fellow of Trinity College, Cambridge, but he lost this Fellowship as he could not make the declaration at that time required from persons proceeding to the M.A. degree. It was but a year or two before his death that he took this degree. In consequence of the extreme delicacy of his health, he passed many of his winters abroad; and, we presume, in these intervals acquired the extensive acquaintance with foreign languages which he undoubtedly possessed. He mastered the mathematical theories of Logic put forth by Boole, Jevons, and others, and in this direction was occasionally of service to the Society. It is evident, from his communications to the *Educational Times* and other journals, that he took great interest in the philosophy and logic of the Theory of Probabilities. A review of Mr. Venn's "Logic of Chance," written by him, is considered to be a very able one. He appears to have been ever ready to assist any applicant to him for information on the subjects which we have above said more especially occupied his attention. Letters from him on these and kindred subjects have been printed in *Nature*, (*cf. Proc.*, Vol. xi., pp. 121, 156, and elsewhere), *Mind*, and the *Academy;* see also Mrs. De Morgan's Memoir of A. de Morgan (pp. 158, 159). From "The Life of Clerk Maxwell" we get some insight into Monro's intellectual stature, as in this work are printed several letters which passed between these two able men. Some of these (p. 346) treat of communications made to the London Mathematical Society. Mr. Monro, when at Cambridge, read for a time with Mr. Todhunter, who had the very highest opinion of his ability. The Rev. J. Venn, who wrote a short notice of Monro in the *Cambridge Gazette* (in which he dwells "mostly on his wonderfully versatile powers, and his lavish assistance to his friends in the way of revising proof sheets and offering other critical aid"), has put the following remarks, penned in the midst of pressing engagements, at our service:—"I did not get to know him personally until years after he left college; before then I had only heard of him as a man with much more of the real taste and capacity for mathematics than we commonly find among students here, but of a desultory habit, and only devoting his spare hours to that pursuit." Mr. Venn then writes,

"His most elaborate paper,—and one which was more purely mathematical than any other which I saw of his,—was one contributed by him to the Cambridge Philosophical Society (through De Morgan), but not published, nor, I believe, read there. It is long since I saw it,—in fact, he wrote it some sixteen years ago; but it struck me as remarkably able. It was on Probability, and consisted in part of some criticism of views which I had expressed, but going much more deeply into the mathematics of the subject, and written with all his usual point and terse humour.* It always seemed to me that his special bent was towards those outlying facts on the boundaries of mathematics on which alone I had communication with him; and that, if he had had better health, and the power of sustained work which might have come of that, he would have probably done more in those directions than any recent writer, except Boole."

The following is, we believe, a complete list of Mr. Monro's mathematical papers:—

"An Example of the Instinct of Constructive Geometry," *Quar. Jour. of Math.*, ii., 1858, pp. 225—229.

"On the Nomenclature of the Physical Sciences," *Phil. Mag.*, xxviii., 1864, pp. 461—465.

"On a case of Stereoscopic Illusion (1864)," *Phil. Mag.*, xxix., 1865, pp. 15—19.

"On an Integral expressing the 'range' of Conditioned Variables," *Quar. Jour. of Math.*, viii., 1867, pp. 278—283.

"On the Colour Yellow," *Nature*, iii., 1871, p. 246.

"Baltzer on the Number of Terms in a Determinant with a Vanishing Diagonal (1872)," *Mess. of Math.*, ii., 1873, pp. 38, 39.

"Note on the Inversion of Bernoulli's Theorem in Probabilities," *Proc. Lond. Math. Soc.*, v., pp. 74—78, 145, 146 (read Feb. 12th, 1874).

"On Flexure of Spaces," *Proc. Lond. Math. Soc.*, ix., pp. 171—176. [In correspondence about this paper, Mr. Monro kindly supplied some additional references to German and Russian writers on "what we may call Fairy Space." These were not used, in consequence of Mr. Halsted's Bibliography having appeared in the *American Jour. of Math.* (see *Proc.*, ix., p. 275).

Mr. Monro was elected a member May 11th, 1871.

Mr. Charles Robert Rippin, formerly of St. John's College, Cambridge, took his B.A. degree in 1861, when he was placed 29th on the list of Wranglers, and proceeded M.A. in 1866. He was elected a Member of this Society, Oct. 16th, 1865.

Prof. Enoch B. Seitz, M.A., was born August 24th, 1846, on a farm in Fairfield County, near Lancaster, U.S.A., where he passed his boyhood. He received in his youth the elements only of a common school education. He afterwards pursued a mathematical course two years in the Ohio Wesleyan University, then taught in the High School,

* This paper we are hopeful of seeing through the kind offices of Mr. C. H. Monro.

Greenville, Ohio, from 1872 till 1879. In this latter year he was elected Professor of Mathematics in the North Missouri State Normal School, at Kirksville, Missouri, "where he won golden opinions from every individual who formed his acquaintance." He died Oct. 8th, 1883, of typhoid fever. His favourite branches were "Average" and "Probability," and in these subjects he proposed and solved numerous questions in our own *Educational Times*, and in most of the American mathematical journals. Numerous testimonies as to his ability and amiability are published in *The Mathematical Magazine*, Erie, Pa. (Vol. i., No. 8), and in *The School Visitor*, Ansonia, Ohio (Vol. iv., No. 11, pp. 210—214).* Prof. Seitz was elected a member March 11th, 1880, being the fifth American mathematician who has joined our ranks.

March, 1884. R. T.

---

* Mr. Artemas Martin, to whose courtesy we are indebted for copies of these journals, writes:—"His name is written in unfading characters on the *pinnacle* of the Temple of Fame."

INDEX.

INDEX OF PAPERS, &c.

[*The mark * indicates that the Paper has either been printed in extenso, or an abstract of it given in the Proceedings or the Appendix.*]

CAYLEY, A., F.R.S.

*On the Automorphic Transformation of a Binary Cubic Function, pp. 103—108.
*On Monge's Mémoire sur la théorie des Déblais et des Remblais, pp. 139—142.
*On Mr. Wilkinson's Rectangular Transformation, pp. 222—229.
*Addition to Paper by Mr. J. Hammond, pp. 88—91.

COCKLE, Sir JAMES, F.R.S.

*On the Explicit Integration of Certain Differential Resolvents, pp. 18—22.

ELLIOTT, E. B., M.A.

*On Certain Relations between Volumes of Loci of Connected Points, pp. 62—71.

FORSYTH, Prof., M.A.

*On In- and Circumscribed Polyhedra, pp. 35—45.
On the Normal Integrals connected with Abel's Theorem, p. 143.

GENESE, Prof., M.A.

*Relations between the Common Points and Common Tangents of Two Conics, pp. 235—237.

GLAISHER, J. W. L., F.R.S.

*Calculation of the Hyperbolic Logarithm of π, pp. 134—139.
A Differential Equation allied to Riccati's, p. 103.

GRIFFITHS, J., M.A.

*Note on the Derivation of Elliptic Function Formulæ from Confocal Conics, pp. 46—48.
*A Method for Reducing the Differential Expression

$$dt \,/\sqrt{\{t-\alpha\,.\,t-\beta\,.\,t-\gamma\,.\,t-\delta\}},$$

to the Standard Form, pp. 196—198.

HAMMOND, J., M.A.

*Note on an Exceptional Case in which the Fundamental Postulate of Prof. Sylvester's Theory of Tamisage fails, pp. 85—88.
*On the Use of Certain Differential Operators in the Theory of Equations, pp. 119—129.
Remarks on a recent Paper by Prof. Sylvester, F.R.S., in the *American Journal of Mathematics*, p. 133.

HART, H., M.A.

*On the Sylvester-Kempe Quadruplane, pp. 199—202.

HILL, Prof. M. J. M., M.A.

*Calculation of the Equation which Determines the Anharmonic Ratios of the Roots of a Quintic, pp. 179—196.

HIRST, Dr., F.R.S.

The Resolution of Congruences into Systems of Quadric Reguli, p. 103.
*Cremonian Congruences, pp. 259—301.

JEFFERY, H. M., F.R.S.

*On Certain Quartic Curves which have a Cusp at Infinity, whereat the Line at Infinity is a Tangent, pp. 71—85.
*On Bicircular Quartics with Collinear Foci, pp. 239—258.

JENKINS, M., M.A.

*Note on Captain MacMahon's Paper, p. 132.

LAMB, Prof. H., M.A.

*On the Vibrations of a Spherical Shell, pp. 50—56.
*On the Mutual Potential of Two Lines in Space, pp. 301—307.

MACMAHON, Capt. P. A., R.A.

*A Generalization of the Nine Points Properties of a Triangle, pp. 129—132.

RAYLEIGH, Lord, F.R.S.

*Investigation of the Character of the Equilibrium of an Incompressible Heavy Fluid of Variable Density, pp. 170—177.

ROBERTS, R. A., M.A.

*On Unicursal Twisted Quartics, Pt. i., pp. 22—34; Pt. ii., pp. 308—314.
*On Polygons circumscribed about a Tricuspidal Quartic, pp. 56—62.

ROBERTS, S., F.R.S.

*Remarks on Mathematical Terminology and the Philosophical bearing of recent Mathematical Speculations concerning the Realities of Space, pp. 5—15.
*On Curves obtained by an Extension of Maclaurin's Method of constructing Conics, pp. 110—118.

ROBERTS, W. R. W., M.A.

*On the Motion of a Particle on the Surface of an Ellipsoid, pp. 230—235.

ROUTH, E. J., F.R.S.

*On Simultaneous Differential Equations, with special reference to (1) The Roots of the Fundamental Determinant, (2) The Method of Multipliers, pp. 144—162.
*Some Remarks on those Solutions of Linear Differential Equations which depend on Multiple Types, pp. 162—170.

SCOTT, R. F., M.A.

*On Compound Determinants, pp. 91—102.

SMITH, Prof. H. J. S., F.R.S.

On the absolutely Least Periods of the Elliptic Functions, p. 49.

Spottiswoode, W., P.R.S.
*Note on Quartic Curves in Space, pp. 15—18.

Tucker, R., M.A.
*On Two Concentric Circles, pp. 202, 316—321.

Walker, J. J., F.R.S.
Geometrical Proof of Griffiths's Extension of Graves's Theorem, p. 49.

Wilkinson, Rev. M. M. U., M.A.
*On Elliptic Function Formulæ connected with the Transformation of Rectangular Coordinates, pp. 203—222.

Wolstenholme, Prof., M.A., Sc.D.
*Equations of the Loci of the Intersections of Three Tangent Lines and of Three Tangent Planes to any Quadric $u = 0$, pp. 177, 178.

GENERAL INDEX.

Académie des Sciences, Comptes Rendus, 269, 288.
Académie, Mémoires de l', 139.
Album presented, 49.
Auditor, 1, 109.
Authors quoted:—
Burckhardt, 134.
Casey, 239, &c., 246, 255.
Cayley, 47, 48.
Chasles, *Ap. Hist.*, 255, 256.
Cremona, 262.
Dase, 135.
Everett, 56.
Kulik, 137.
Legendre, 212.
Maxwell, Electricity, 302.
Rayleigh, 52.
Reid, Dr., 13.
Reiss, 94.
Routh, 144, 146.
Salmon, Dr., Higher Algebra, 24, 86, 126, 181, 184, 194, 196.
Salmon, Dr., Solid Geometry, 45, 230, 232, 234, 313.
Salmon, Dr., H. P. Curves, 61, 71, 110, 254.
Schubert, 263.
Schulze, 135.
Serret, 103.
Steiner, 266, 268.
Thomson and Tait, 54, 144.
Vega, 135.
Wolfram, 135.
Berlin, Abhandl. der K. Ac. d. W. zu, 269.
Bologna, Trans. of Institute of, 259.
British Association, Report of, 18.
Brocard, 316—321.
Caporali, 285, 288.
Casey, 201, 319.
Cayley, 15, 17, 30, 35, 38, 61, 85, 86, 88, 119, 126, 202, 256, 270.
Circle, "Triplicate-Ratio," def.: properties and reff., 316—321.
Clifford, 14.
Cockle, 18.
Council, Election of, 3, 143.
Crelle, 94, 262, 266, 268, 274, 302.
Cremona, 259, 270, 290.
Cremonian Correspondence, 259.
,, Congruence, 259.
Crofton, 253.
Darboux, 288.
De Morgan Memorial, 5.
Diary, Lady's and Gentleman's, 318.
D'Ocagne, 320.
École Polytechnique, Journal de l', 94.
Educational Times, 317.
Folium, 75.
Führmann, 320.
Füss, 35.
Göttingen Nachrichten, 263.
Griffiths, 47.
Hammond, 119, 122.
Harley, 18.
Hart, 239, 241.
Helmholtz, 302.
Hirst, 282.
Jacobi, 35, 320.
Jeffery, 71, 246, 251.
Jenkins, 131.
Jevons, 170.
Journals presented, 3.
Journals subscribed for, 3.

Kant, 14.
Kempe, 63, 316.
Kiehl, 321.
Klein, 263.
Kummer, 262, 269, 271, 290.
Lamb, 50.
Lemoine, 320.
Le Paige, 97.
Leudesdorf, 62.
Library, presents made to, 4, 49, 103, 109, 133, 143, 202, 238, 314.
Lie, 263.
Liguine, 316.
Lincei, R. A. dei, 285, 288.
Liouville, 61, 256.
Lobatchewsky, 13.
Maclaurin, 111.
MacMahon, 131.
Mathematical Journal, Camb. and Dub., 94.
Mathematical Society of Edinburgh, 316.
Mathematical Society, Proceedings of, 18, 38, 50, 56, 71, 119, 122, 146, 216, 239, 241, 242, 245, 246, 251, 253, 282, 308, 316.
Mathematics, American Journal of, 86, 96.
Mathematics, Messenger of, 62, 63, 96, 98, 131.
Mathematics, Quarterly Journal of, 202, 316, &c.
Mathematics, Reprint from *Educational Times*, 114, 316.
Mathématiques, Nouvelle Correspondance de, 320.
Mathématique de France, Bulletin de la Société, 97.
Mathématiques et Astr., Bulletin des Sciences, 259—261, 270, 290, 292, 295, 298, 316.
Mathématiques, Nouvelles Annales de, 269, 320.
Mathematische Annalen, 75.
Mathesis, 320.
Members, Election of, &c., 3, 49, 103, 109, 133, 202, 237.
Members, Number of, 1.
Minding, 35.
Monge, 139.
Muir, 94, 96.
Nature, 321, 322.
Neuberg, 320.
Notation, 8.
Obituary, 1, 3, 133.
——— Notices, Dr. Spottiswoode, 321.
——— ——— Prof. H. J. S. Smith, 322.
——— ——— Mr. C. J. Monro, 323.
——— ——— Mr. C. R. Rippin, 323.
——— ——— Prof. E. B. Seitz, 323.
Papers read, 1-4, 49, 103, 109, 133, 143, 202, 237.
Philosophical Magazine, 30, 35, 94, 170, 316.
——— Transactions, 35, 196.
Picquet, 94, 95.
"Proceedings," Receivers of, 315.
Progress of the Society, 5.
Quadruplane, 199.
Quetelet, 255.
Rawson, 18.
Rayleigh, 56.
Reguli, 270.
Reports, Treasurer's and Secretaries', 1-3.
Reye, 266, 268.
Richelot, 35.
Riemann, 12.
Roberts, R. A., 308.
Roberts, S., 199, 201, 316.
Routh, 146.
Royal Society of Victoria, Transactions of, 18.
Schröter, 294.
Scott, Miss C. A., 317.
Scrutators, 3.
Smith, H. J. S., 216.
Steiner, 35.
Style, Mathematical, 7.
Sylvester, 30, 94, 196.
Tait, 316.
Thanks, to Treasurer, 49.
——— to Auditor, 109.
Treasurer, Resignation of, 49.
——— Election of, 49.
Van Swinden, 320.
Weber, 304.
Zeuthen, 75.

ERRATUM.

By a printer's error, "Dr." is printed for "Mr." Horace Lamb on top of pp. 302, 304.

Printed by C. F. Hodgson & Son, 1 Gough Square, Fleet Street.

VOLUMES ALREADY ISSUED.

Vol.	I. (from January 1865 to November 1866)	10*s.*
,,	II. (from November 1866 to November 1869)	16*s.*
,,	III. (from November 1869 to November 1871)	20*s.*
,,	IV. (from November 1871 to November 1873)	31*s.* 6*d.*
,,	V. (from November 1873 to November 1874)	15*s.*
,,	VI. (from November 1874 to November 1875)	21*s.*
,,	VII. (from November 1875 to November 1876)	21*s.*
,,	VIII. (from November 1876 to November 1877)	21*s.*
,,	IX. (from November 1877 to November 1878)	21*s.*
,,	X. (from November 1878 to November 1879)	18*s.*
,,	XI. (from November 1879 to November 1880)	12*s.* 6*d.*
,,	XII. (from November 1880 to November 1881)	16*s.*
,,	XIII. (from November 1881 to November 1882)	18*s.*

BIBL.
LONDIN.
UNIV.